93. *Pseudo-compactness and μ-convergence*

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Following E. Hewitt, a completely regular space is said to be *pseudo-compact*, if every real continuous function on it is bounded. In a paper [2] of K. Iséki a characterisation of pseudo-compact spaces was given in terms of uniformly convergences and its related concepts. In this Note, we shall give a further characterisation by the μ-convergence of G. Sirvint [6].

A sequence $\{f_n(x)\}$ of real functions defined in an abstract set S is said to be *quasi-uniformly convergent* to $f(x)$ on S if it converges to $f(x)$ and if, for given positive ε and integer N, there is a finite number of indices $n_1, n_2, \cdots, n_k \geq N$ such that for each x of S, at least one of the following relations holds:

$$|f_{n_i}(x)-f(x)|<\varepsilon, \qquad i=1, 2, \cdots, k.$$

Then we have the following

Lemma 1. If a sequence $\{f_n(x)\}$ of real continuous function of a pseudo-compact space S is convergent to 0, *then it converges quasi-uniformly to* 0 *on S.*

Proof. For a given positive ε and a given integer N, we shall define the sets O_n as

$$O_n=\{x\,|f_n(x)|<\varepsilon\}, \qquad n=N, N+1, \cdots .$$

Since $f_n(x)$ is continuous, each $\{O_n\}$ is open, and from $f_n(x)\to 0$ on S, $\{O_n\}$ is a countable open covering of S. Therefore, by a theorem of S. Mardešić and P. Papić [5], there is a finite set of indices $n_1, \cdots, n_k$ such that $\bigcup_{i=1}^{k} \overline{O}_{n_i} \supset S$. Therefore $f_n(x)$ converges quasi-uniformly to 0.

Conversely, we shall show the following

Lemma 2. Let S be a completely regular space. If a sequence $\{f_n(x)\}$ of continuous functions on S which converges to 0 *converges quasi-uniformly to* 0, *then S is pseudo-compact.*

Proof. Suppose that there is an unbounded continuous function $f(x)$ on S. Then we can find a sequence $\{x_n\}$ such that $x_n \in S$ and $|f(x_n)|\to\infty$ $(n\to\infty)$. Let $f_n(x)=\dfrac{f(x)}{f(x_n)}$, then we have $f_n(x)\to 0$ $(n\to\infty)$ on S (pointwise convergence!). For a given N and $N<m$, we have

$$|f_{n_1}(x_{n_2})|=\left|\frac{f(x_{n_2})}{f(x_{n_1})}\right|>1$$

for $N\leq n_1\leq m<n_2$. Since m may be taken arbitrary, this implies that