# 89. A Theorem on Continuous Convergence

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A few years ago some interesting results on continuous convergence were obtained by C. Kuratowski [3], R. Arens and J. Dugundji [1]. Following H. Hahn [2], we shall define it as follows: A sequence of real valued functions $f_n(x)$ on a topological space $S$ *converges continuously* to $f(x)$ on $S$ if and only if $f_n(x_n)\to f(x)$ whenever $x_n\to x$ on $S$. In his "General Topology", W. Sierpiński [4, pp. 156-158] has proved that *a metric space M is compact if and only if the continuously convergence on M implies uniformly convergence on M.* Recently, S. Stoilow [5] proved a theorem on continuous convergence. In this Note, we shall generalize the theorem mentioned above of W. Sierpiński. To do so, we shall first prove the following

*Theorem 1. If $f_n(x)$ converges continuously to $f(x)$ on a sequentially compact space $S$, the convergence is uniform.*

By a *sequentially compact space,* we shall mean every sequence has a convergent subsequence.

Following the method of S. Stoilow [5, pp. 247-248], if $f_n(x)$ on any topological space $S$ converges continuously to $f(x)$, we can prove that $f(x)$ is sequentially continuous: $x_n\to x_0$ implies $f(x_n)\to f(x_0)$. To prove Theorem 1, suppose that $f_n(x)$ is not convergent uniformly to $f(x)$. Then we can find a positive $\varepsilon$ and an infinite sequence $x_n$ such that

$$|f_{m_n}(x_n)-f(x_n)|>\varepsilon. \quad (n=1,2,\cdots)$$

Since $S$ is sequentially compact, there is a convergent subsequence $x_{n_i}$ of $x_n$. Let $x_0$ be its limit point, then we have $f(x_{n_i})\to f(x_0)$. On the other hand, since $f_{m_{n_i}}(x)$ converges continuously to $f(x)$, we have $f_{m_{n_i}}(x_{n_i})\to f(x_0)$, which contradicts $|f_{m_{n_i}}(x_{n_i})-f(x_{n_i})|>\varepsilon$. This completes the proof.

Next, we shall prove the following

*Theorem 2. If the continuous convergence of $f_n(x)$ on a completely regular space $S$ to $f(x)$ on $S$ implies the uniformly convergence, then $S$ is countably compact.*

Proof. Suppose that $S$ is not countably compact, then there is a countable set $a_n$ without cluster point. Therefore for each $a_n$, there is a neighbourhood $U_n$ of $a_n$ such that $U_m\frown U_n=\phi$ for $m\neq n$. Since $S$ is completely regular, we can find a continuous function $f_n(x)$ such that