112. On Operators of Schaefer Class in the Theory of Singular Integral Equations

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The purpose of the present brief note is to observe the theorems of H. Schaefer [4] under the method of J. W. Calkin [1] and A. F. Ruston [3].

1. Let H be a separable Hilbert space. A bounded linear operator a acting on H will be called an operator of *Schaefer class*, or a σ-transformation in Schaefer's sense, if a satisfies

1) the range R, or $R(a)$, is closed and has finite codimension,

2) the null-space N, or $N(a)=\{\xi;\, \xi a=0\}$, is finite dimensional.

Since $\operatorname{codim} R=\dim H/R=\dim N^*$, where N^* is the null-space of the adjoint a^* of a, 1) is equivalent to assume that the range is closed and N^* has finite dimension.

The typical examples of operators of Schaefer class are

I. if c is completely continuous, then $1-c$ is an operator of Schaefer class by the Riesz theory,

II. if d is regular, i.e. d has the bounded inverse d^{-1}, then d is of Schaefer class.

The set S of all operators of Schaefer class is self-adjoint in the sense that $a\in S$ implies $a^*\in S$ [4, Satz 1].

2. Let B be the B^*-algebra of all bounded linear operators acting on H, and let C be the ideal of B consisting of all completely continuous members. The natural homomorphism of B onto the quotient algebra B/C will be denoted by $\sharp$. The aim of the present note is to show

THEOREM. *An operator a is of Schaefer class if and only if a is regular modulo C,* i.e. *$a^\sharp$ has an inverse in B/C.*

3. The proof of the sufficiency is contained essentially in [4, §2 Hilfssatz]. Let $ab\equiv ba\equiv 1 \mod C$, that is, $ab=1-c$ with $c\in C$ and $ba=1-c'$ with $c'\in C$. By the well-known theory of Riesz, $N(ab)$ and $N^*(ba)$ are finite-dimensional, whence N and N^* are also. Thus it remains to show that Ha is closed. Again, by the Riesz theory, $1-c$ gives one-to-one bicontinuous (isomorphic) mapping of a (closed) subspace F onto another. Therefore, Fa is closed and so Ha is closed too, since F has finite-codimension.

The proof of necessity is same as that of [4, Satz 12]. By the assumption a gives an isomorphism of $N^\perp$ onto R, whence the inverse b' of a on R exists and is bounded by a theorem of Banach. If p is