110. On Complete Orthonormal Sets in Hilbert Space

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It is well known that a set in a separable Hilbert space*) is complete, if the set is sufficiently near a complete orthonormal set under some additional conditions. Such theorems were obtained by Paley, Wiener [7], Bellman [3] and Pollard [8] in United States, and Bary [1, 2], Kostyučenko and Skorohod [6] in Soviet Russia, and Hilding [4, 5] in Sweden.

Kostyučenko and Skorohod have given a simple proof of Bary theorem: if $\{\varphi_n\}$ and $\{\psi_n\}$ are orthonormal systems in Hilbert space, and if $\sum_{n=1}^{\infty}\|\varphi_n-\psi_n\|<\infty$, then both systems are complete, if one is. S. H. Hilding [4] has shown that, *if* $\{\varphi_n\}$ *is a complete orthonormal system and if* $\sum_{n=1}^{\infty}\|\varphi_n-\psi_n\|<1$, *then* $\{\psi_n\}$ *is complete,* and he has also obtained other two results; *let* $\{\varphi_n\}$ *be a complete orthonormal system, and let* $r_n=\|\varphi_n-\psi_n\|$,

1) *if* $\|\psi_n\|=1$ *for* $n=1, 2,\cdots$ *and if* $\sum_{n=1}^{\infty}r_n^2\left(1-\frac{r_n^2}{4}\right)<1$,

or

2) *if* $(\varphi_n, \psi_n)=0$ *and if* $\sum_{n=1}^{\infty}\frac{r_n^2}{1+r_n^2}<1$,

then $\{\psi_n\}$ *is complete.*

We shall prove the following

Theorem. Let $\{\varphi_n\}$ *and* $\{\psi_n\}$ *be two orthonormal systems, let* $r_n=\|\varphi_n-\psi_n\|$.

1) *If* $\{\varphi_n\}$ *is complete,* $\|\psi_n\|=1$ *for* $n=1, 2,\cdots$ *and* $\sum_{n=1}^{\infty}r_n\left(1-\frac{r_n^2}{4}\right)<\infty$, *then* $\{\psi_n\}$ *is complete.*

2) *if* $\{\varphi_n\}$ *is complete,* $(\varphi_n, \varphi_n-\psi_n)=0$ *for* $n=1, 2,\cdots$, *and* $\sum_{n=1}^{\infty}\frac{r_n^2}{1+r_n^2}<\infty$, *then* $\{\psi_n\}$ *is complete.*

To prove Theorem, we shall use the techniques by S. H. Hilding [4], Kostyučenko and Skorohod [6]. First, we shall prove the second part of Theorem. Since the series $\sum_{n=1}^{\infty}\frac{r_n^2}{1+r_n^2}$ converges, there is an integer N such that

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*) For fundamental concepts, see B. Sz-Nagy: Spektraldarstellung linearer Transformationen des Hilbertschen Raumes, Berlin (1942).