109. On Imbedding a Metric Space in a Product of One-dimensional Spaces

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It is well known that every separable metric space can be imbedded in Hilbert cube I^{ω}. Recently K. Morita has proved that a regular space having σ-star-finite basis can be imbedded in the topological product $N(\Omega)\times I^{\omega}$ of a generalized Baire's zero-dimensional space $N(\Omega)$ and I^{ω}.[1] On the other hand the author has shown that every n-dimensional metric space can be imbedded in a product of n+1 one-dimensional spaces.[2] However, it seems that there is little study on imbedding general metric spaces in a product of one-dimensional spaces. The purpose of this note is to show that every metric space can be imbedded in a product of countably many one-dimensional spaces.

In this note we concern ourselves only with metric spaces and mean by a covering an "open" covering.

Lemma 1. *For every covering* $\mathfrak{U}$ *of a metric space* R *there exist collections* $\mathfrak{U}_i$ $(i=1,2,\cdots)$ *of open sets and a covering* $\mathfrak{V}$ *such that* $\mathfrak{V}<\bigcup_{i=1}^{\infty}\mathfrak{U}_i<\mathfrak{U}$ *and such that each* $S^2(p,\mathfrak{V})$ $(p\in R)$ *intersects at most one set of* $\mathfrak{U}_i$ *for a fixed* i *and finitely many sets of* $\bigcup_{i=1}^{\infty}\mathfrak{U}_i$

Proof. As it was shown, for every fully normal space, by A. H. Stone,[3] there exist open collections $\mathfrak{U}_i$ $(i=1,2,\cdots)$ and a covering $\mathfrak{W}$ such that $\mathfrak{W}<\bigcup_{i=1}^{\infty}\mathfrak{U}_i<\mathfrak{U}$ and such that each set of $\mathfrak{W}$ intersects at most one set of $\mathfrak{U}_i$ and finitely many sets of $\bigcup_{i=1}^{\infty}\mathfrak{U}_i$. If we take a covering $\mathfrak{V}$ satisfying $\mathfrak{V}^{\triangle\triangle}<\mathfrak{W}$, then all the conditions of this lemma are satisfied.

Lemma 2. *For every coverings* $\mathfrak{P}_i$ $(i=1,2,\cdots)$ *with order* $\mathfrak{P}_i\leqq 2$ *and* $\mathfrak{V}$ *satisfying* $\mathfrak{V}<\bigwedge_{i=1}^{\infty}\mathfrak{P}_i$, *there exist locally finite coverings* $\mathfrak{N}_i$ $(i=1,2,\cdots)$ *such that* $\mathfrak{N}_i^*<\mathfrak{P}_i$, *order* $N_i\leqq 2$ $(i=1,2,\cdots)$ *and such that there exists a covering* $\mathfrak{W}$ *satisfying* $\mathfrak{W}<\bigwedge_{i=1}^{\infty}\mathfrak{N}_i$.

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1) The proof of this theorem is unpublished. Cf. K. Morita: Normal families and dimension theory for metric spaces, Math. Ann., **128** (1954). Cf. also J. Nagata: On imbedding theorem for non-separable metric spaces, Jour. Inst. Polytech. Osaka City Univ., **8**, no. 1 (1957).

2) Note on dimension theory, Proc. Japan Acad., **32**, no. 8 (1956).

3) A. H. Stone: Paracompactness and product spaces, Bull. Amer. Math. Soc., **54**, no. 10 (1948).