108. *On Topological Properties of W^* algebras*

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1. In this paper, we shall show some topological properties of W^*-algebras and their applications. Main assertions are as follows: (1) *Any closed invariant subspace of the adjoint space of C^*-algebras is algebraically spanned by positive functionals belonging to itself* [Theorem 1]. (2) *The direct product $M_1 \otimes M_2$ of W^*-algebras M_1 and M_2 is purely infinite, whenever either M_1 or M_2 is purely infinite* [Theorem 2]. This second assertion is the positive answer for a problem of J. Dixmier [4], and we can assert that all questions concerning the "type" of the direct product of W^*-algebras are now solved.

2. Let A be a C^*-algebra, $\tilde{A}$ the adjoint space of A. A subspace V of A is said invariant, if $f \in V$ means fa, $bf \in V$ for $a, b \in A$, where $\langle x, fa\rangle = \langle xa, f\rangle$ and $\langle x, bf\rangle = \langle bx, f\rangle$, where $\langle x, f\rangle$ is the value of f at $x(\in A)$.

Theorem 1.[1] *Any closed invariant subspace of $\tilde{A}$ is algebraically spanned by positive functionals belonging to itself.*

Proof. Let $\tilde{\tilde{A}}$ be the second adjoint space of A, then by Shermann's theorem (cf. [10]) $\tilde{\tilde{A}}$ is considered a W^*-algebra and A is a C^*-subalgebra of $\tilde{\tilde{A}}$, when it is canonically imbedded into $\tilde{\tilde{A}}$ as a Banach space.

Let V^0 be the polar of V in $\tilde{\tilde{A}}$, that is, $V^0 = \{a \mid |\langle a, f\rangle| \leqq 1, a \in \tilde{\tilde{A}}$ and all $f \in V\}$, then it is a $\sigma(\tilde{\tilde{A}}, \tilde{A})$-closed ideal of A, for $|\langle bac, V\rangle| = |\langle a, bVc\rangle| = |\langle a, V\rangle| \leqq 1$ for $a \in V^0$ and $b, c \in A$; hence $bac \in V^0$. Since A is $\sigma(\tilde{\tilde{A}}, \tilde{A})$-dense in $\tilde{\tilde{A}}$ and V^0 is $\sigma(\tilde{\tilde{A}}, \tilde{A})$-closed, this means $bac \in V^0$ for $b, c \in \tilde{\tilde{A}}$, so that V^0 is an ideal.

On the other hand, by a classical theorem of Banach spaces, the adjoint space of V is considered the quotient space $\tilde{\tilde{A}}/V^0$. Since $\tilde{\tilde{A}}/V^0$ is a C^*-algebra, by the author's theorem [8, 9] it is a W^*-algebra and $\sigma(\tilde{\tilde{A}}/V^0, V)$ is the σ-weak topology of $\tilde{\tilde{A}}/V^0$, that is, composed of all σ-weakly continuous linear functionals on $\tilde{\tilde{A}}/V^0$; hence by Dixmier's theorem [3] V is algebraically spanned by positive functionals belonging to itself. Moreover, since the positiveness of elements of V on $\tilde{\tilde{A}}/V^0$ means the positiveness on A, this completes the proof.

Now, let ν be a measure on a measure space and $L^1(\nu)$ and $L^\infty(\nu)$