107. On Generalized Walsh Fourier Series. I

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1. We shall state some theorems on generalized Walsh Fourier series, that is, on Fourier series with respect to the system of the generalized Walsh functions.

Let $\{\alpha(n)\}$ be a sequence of integers not less than 2, and put $A(0)=1$, $A(n)=\alpha(0)\alpha(1)\cdots\alpha(n-1)$, $A(-n)=1/A(n)$. The "generalized Rademacher functions" $\phi_n(t)$ $(n=0,1,2,\cdots)$ are defined as

$$\phi_n(t)=\exp(2\pi ik/\alpha(n))$$

for t belonging to the left-semiclosed intervals

$$(kA(-n-1),\ (k+1)A(-n-1)) \qquad k=0,1,\cdots,A(n+1)-1$$

and $\phi_n(t+1)=\phi_n(t)$ for all t.

Now we can define the "generalized Walsh functions" $\psi_n(t)$ $(n=0,1,2,\cdots)$. Let

$$\psi_0(t)=1$$

and for $n\geqq 1$,

$$\psi_n(t)=\phi_{n(1)}^{a(1)}(t)\,\phi_{n(2)}^{a(2)}(t)\cdots\phi_{n(r)}^{a(r)}(t)$$

provided that n is expressed in the form

$$n=a(1)A(n(1))+a(2)A(n(2))+\cdots+a(r)A(n(r))$$

where

$$n(1)>n(2)>\cdots>n(r)\geqq 0;\ 0<a(j)<\alpha(j) \quad (j=1,2,\cdots,r).$$

The functions $\psi_n(t)$ thus defined form a complete orthonormal system over the interval $(0,1)$. If $\alpha(n)=2$ $(n=0,1,2,\cdots)$, the system reduces to that of Walsh, and the case $\alpha(n)=\alpha$ was studied by H. E. Chrestenson [1]. The general definition seems to have been given by J. J. Price (cf. [7]), but we have not been able to know the details.

We assume in the sequel that, unless others are stated explicitly, the sequence $\{\alpha(n)\}$ is bounded, say $\alpha(n)\leqq\alpha$ $n=0,1,2,\cdots$. Though this assumption may seem stringent, it is necessary, in order to obtain positive results, to confine the "growth" of $\alpha(n)$ under a certain restriction (see Theorem 4 below).

2. The key theorem in the $L^p(p>1)$ theory of Walsh Fourier series is the following, due to R. E. A. C. Paley [6]:

Theorem P. *Let* $f(t)\in L^p(0,1)$, $f(t)\sim\sum_{n=0}^{\infty}c_n\psi_n(t)$. *Then, putting* $f_n(t)=\sum_{\nu=2^n}^{2^{n+1}-1}c_\nu\psi_\nu(t)$ $(n=0,1,2,\cdots)$, *one has*