106. On the Continuity of Norms

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Let R be a universally continuous[1] normed semi-ordered linear space. A norm on R is said to be *continuous*, if $a_\nu \downarrow_{\nu=1}^{\infty} 0$[2] implies $\inf_{\nu=1,2,\cdots} \|a_\nu\|=0$. The importance of continuity of a norm is in the fact that every norm-bounded linear functional on R is, roughly speaking, represented by a continuous function on the proper space of R (cf. [3]). In this note, we consider some conditions of the continuity of norms on R. We use the terminologies and notations in [4].

H. Nakano obtained the following three conditions of continuity:

Theorem A. *If every norm-bounded linear functional on R is continuous,*[3] *the norm is continuous* [4, Theorem 31.10].

Theorem B. *If a norm on R is separable and semi-continuous,*[4] *it is continuous* [4, Theorem 30.27].

Theorem C. *If a norm on R is uniformly monotone and complete, it is continuous* [4, Theorem 30.22].

In the sequel, the set of a type: $\{x;\ a\leq x\leq b\}$ is called a *segment*.

We know that the semi-continuity implies the completeness of segments [6, Theorem 3.3]. We shall replace semi-continuity of a norm by the completeness of segments of R in proving the continuity of a norm.

A general condition for continuity is contained in

Lemma 1. *A norm on R is continuous, if and only if every segment of R is complete and the norm satisfies the condition:*

(1) $[p_\nu][p_\mu]=0$,[5] $\nu\neq\mu$ $(\nu,\mu=1,2,\cdots)$ *implies* $\lim_{\nu\to\infty}\|[p_\nu]a\|=0$ $(a\in R)$.

Proof (cf. [3, Satz 14.3]). If the norm is continuous, it is semi-continuous, hence every segment is complete. For $a\in R$ and $[p_\nu][p_\mu]=0$, $\nu\neq\mu$ $(\nu,\mu=1,2,\cdots)$, we have $(o)\text{-}\lim_{\nu\to\infty}[p_\nu]a=0$,[6] hence by continuity

1) *Universal continuity* means that for any $a_\lambda\geq 0$ $(\lambda\in\Lambda)$ there exists $\bigcap_{\lambda\in\Lambda} a_\lambda$.

2) $a_\nu \downarrow_{\nu=1}^{\infty} a$ means that $a_\nu\geq a_{\nu+1}$ $(\nu=1,2,\cdots)$ and $\bigcap_{\nu=1}^{\infty} a_\nu=a$.

3) A linear functional $\tilde{a}$ on R is said to be *continuous* (resp. *universally continuous*), if for any $a_\nu\downarrow_{\nu=1}^{\infty}0$ (resp. $a_\lambda\downarrow_{\lambda\in\Lambda}0$) $\inf_{\nu=1,2,\cdots}|\tilde{a}(a_\nu)|=0$ (resp. $\inf_{\lambda\in\Lambda}|\tilde{a}(a_\lambda)|=0$).

4) A norm is said to be *semi-continuous*, if $0\leq a_\nu\uparrow_{\nu=1}^{\infty}a$ implies $\sup_{\nu=1,2,\cdots}\|a_\nu\|=\|a\|$.

5) $[p]$ is a projection operator to the normal manifold generated by p: $[p]a=\bigcup_{\nu=1}^{\infty}(\nu|p|\frown a)$ for $0\leq a\in R$.

6) (o)-lim means order-limit.