105. *Pseudo-compactness and Strictly Continuous Convergence*

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In my Note [3], the present author gave a characterisation of countably compactness of weakly separable completely regular spaces by using the notion of continuous convergence. In this paper, we shall give necessary and sufficient conditions for pseudo-compactness of completely regular spaces.

First, we shall define two types of the convergences of functions on a topological spaces. One of them (see Definition 2) is due to C. Kuratowski [4]. Next, we note a trivial relation between these two notions for the completeness, and we shall treat some related topics. Finally, we shall state the main Theorems 2 and 3.

Definition 1. *A sequence $\{f_n(x)\}$ on a topological space S is said to be continuous convergent to $f(x)$, if $x_n \to x$ implies $f_n(x_n) \to f(x)$.*

Definition 2. *A sequence $\{f_n(x)\}$ on a topological space S is said to be strictly continuous convergent to $f(x)$, if the convergence of $\{f(x_n)\}$ implies the convergence of $\{f_n(x_n)\}$ and $\lim f_n(x_n) = \lim f(x_n)$.*

Let $f_n(x)$ and $f(x)$ be continuous functions and suppose that $f_n(x)$ is strictly continuous convergent to $f(x)$. If $x_n \to x$ then $f(x_n) \to f(x)$ by the continuity of $f(x)$. Hence, since f_n is strictly continuous convergent, $\lim f_n(x_n) = \lim f(x_n) = f(x)$. Therefore $f_n(x)$ converges to $f(x)$ continuously.

Conversely, let S be a sequentially compact, and suppose that $f_n(x)$ converges continuously to $f(x)$, then we shall show that $f_n(x)$ is strictly continuous convergent to $f(x)$.

To show it, we shall suppose that f_n is not strictly continuous convergent to f, there is a sequence $\{x_n\}$ such that $f(x_n)$ is convergent to a number α and $f_n(x_n)$ is not convergent to α. Therefore we can find a subsequence $f_{n_i}(x_{n_i})$ such that every subsequence of it does not converge to α. Since S is sequentially compact, $\{x_{n_i}\}$ contains a convergent sequence $\{x_{n_{i_j}}\}$. Let x be the limit point of $\{x_{n_{i_j}}\}$. From the hypothesis, $\lim f_{n_{i_j}}(x_{n_{i_j}}) = f(x)$. By the continuity, we have $f(x_{n_{i_j}}) \to f(x)$ and $\lim f(x_{n_{i_j}}) = \lim f(x_n) = \alpha$. Hence we have $f(x) = \alpha$. This shows that $f_{n_{i_j}}(x_{n_{i_j}}) \to \alpha$, which is contradiction.

Next, we shall consider a completely regular space S, and suppose that every sequence of continuous functions which is convergent continuously is strictly continuous convergent. Then we shall show that S is countably compact. Suppose that S is not countably compact, then there is a countable set $\{a_n\}$ without cluster point. Therefore, for