104. *On Homomorphisms of the Ring of Continuous Functions onto the Real Numbers*

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Let X be a C^∞-manifold and $F(X)$ be the ring of all the C^∞-functions on X, or let X be a Q-space[1] and $C(X)$ be the ring of all the real-valued continuous functions on X. Then for a non-trivial homomorphism ϕ (i.e. $\phi(f)\not\equiv 0$) of the function ring $F(X)$ or $C(X)$ into the real number field R, there exists one and only one point p of X such that $\phi(f)=f(p)$ for any f of the respective function ring. Hence it follows that C^∞-manifolds X and Y are differentiably homeomorphic if $F(X)$ and $F(Y)$ are isomorphic,[2] and that Q-spaces X and Y are homeomorphic if $C(X)$ and $C(Y)$ are isomorphic.[3] In this paper we shall study the generalizations of these results. For brevity we use the word 'homomorphism' in place of the word 'non-trivial homomorphism'.

Let X be a completely regular space and let $C(X, R)$ be the ring of all the real-valued continuous functions on X. We denote by $\mathfrak{C}$ a subring of $C(X, R)$ satisfying the following conditions:

(1) $R\subset\mathfrak{C}$,

(2) for a closed set F of X and a point $p\notin F$, there exists a function f of $\mathfrak{C}$ such that $f(p)>\sup_{x\in F} f(x)$,

(3) if $f(x)>a>0$ and $f(x)\in\mathfrak{C}$, then $f^{-1}(x)\in\mathfrak{C}$.

The conditions (2) and (3) are weaker than the following conditions (2′) and (3′) respectively:

(2′) for a closed set F of X and a point $p\notin F$, there exists a function f of $\mathfrak{C}$ such that $0\leqq f(x)\leqq 1$, $f(p)=1$, and $f(x)=0$ if $x\in F$,

(3′) if $f(x)>0$ and $f(x)\in\mathfrak{C}$, then $f^{-1}(x)\in\mathfrak{C}$.

It is obvious that the conditions (1), (2′) and (3′) are all fulfilled, if $\mathfrak{C}=F(X)$ or $C(X)$.

We now define a uniform structure gX of X by the following uniform neighborhoods:

$U_{f_1,\cdots,f_n;\varepsilon}(x)=\{y \mid |f_i(y)-f_i(x)|<\varepsilon,\ i=1,2,\cdots,n\}$, where $f_i\in\mathfrak{C}$ $(i=1,2,\cdots,n)$ and ε is an arbitrary positive number. Then it is easily seen that gX agrees with the topology of X by virtue of (2).

1) By a C^∞-manifold we mean a separable C^∞-manifold. For the definition of a Q-space see [3, 4, 7].

2) See [1].

3) See [3, Theorem 57].