129. A Relation between Two Realizations of Complete Semi-simplicial Complexes

By Yukihiro KODAMA

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1. Let $S(X)$ be a singular complex of a topological space X. J. B. Giever [2] constructed a polytope $P(X)$ which is a geometric realization of $S(X)$ and whose homotopy groups are isomorphic to those of X. Let K be a complete semi-simplicial (c.s.s.) complex [1]. S. T. Hu [3] constructed a polytope $P(K)$, which is a geometric realization of K. Hu's realization associated with $S(X)$ is homeomorphic to Giever's realization $P(X)$. J. Milnor [6] has defined a geometric realization $|K|$ of K which is different from that used by Giever and Hu. In this note, we shall show that Milnor's realization $|K|$ has the same homotopy type as Giever-Hu's realization $P(K)$.

2. Let K be a c.s.s. complex. The face and degeneracy maps of K are transformations such that

$$F_i: K_q \to K_{q-1},\quad q>0,\ i=0,1,\cdots,q,$$
$$D_i: K_q \to K_{q+1},\quad q\geqq 0,\ i=0,1,\cdots,q,$$

where K_q is the set of q-simplexes of K, and satisfy the following commutation rules:

$$\text{(A)}\qquad \begin{gathered} F_iF_j=F_{j-1}F_i,\ D_iD_j=D_{j+1}D_i,\ F_iD_j=D_{j-1}F_i,\ i<j, \\ F_jD_j=F_{j+1}D_j=identity,\ \ D_iD_i=D_{i+1}D_i, \\ F_iD_j=D_jF_{i-1},\quad i>j+1. \end{gathered}$$

We denote by $\Delta_q=(0,1,\cdots,q)$ the standard q-simplex. $e_i:\Delta_{q-1}\to\Delta_q$ and $d_i:\Delta_q\to\Delta_{q-1}$ will denote the simplicial mappings defined by

$$e_i(j)=\begin{cases} j, & 0\leqq j<i, \\ j+1, & i\leqq j<q, \end{cases}\qquad d_i(j)=\begin{cases} j, & 0\leqq j\leqq i, \\ j-1, & i<j\leqq q. \end{cases}$$

Form the topological sum $\tilde{K}=\bigcup_q (K_q\times\Delta_q)$ with the discrete topology on K_q. Consider the following relations:

(i) $\qquad (F_is, x)\approx(s, e_ix),\quad s\in K_q,\ x\in\Delta_{q-1},$

(ii) $\qquad (D_is, x)\approx(s, d_ix),\quad s\in K_q,\ x\in\Delta_{q+1}.$

Milnor's realization $|K|$ is the identification space formed by reducing $\tilde{K}$ by the relations (i) and (ii). The following lemma is proved easily.

Lemma 1. *Giever-Hu's realization $P(K)$ is the identification space formed by reducing $\tilde{K}$ by the relation* (i).

By Lemma 1 there exist natural projections $g:\tilde{K}\to P(K)$ and $f: P(K)\to|K|$.

Lemma 2. *For each 0-cell v of $|K|$, $f^{-1}(v)$ is homeomorphic to a CW-complex Q* [7] *such that*