# 127. A Remark on Pseudo-compact Spaces

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Following Prof. E. Hewitt, a completely regular space is said to be *pseudo-compact*, if every real continuous function on it is bounded. The present writer [4, Corollary 2] proved a theorem closely related to some of results by R. G. Bartle [1] and T. Isiwata [5]. In this Note, we shall show that it is a characterisation of pseudo-compact spaces. Let $\beta(S)$ be the Čech compactification of a completely regular space $S$, and let $f(x)$ be a bounded continuous function. Let $f^*(x)$ be the unique extension of $f(x)$ on $\beta(S)$. Then we have the following

*Theorem. Let $S$ be a completely regular space, then the following statements are equivalent:*

(1) *$S$ is pseudo-compact.*

(2) *For every sequence of bounded continuous functions $f_n(x)$ $(n=1,2,\cdots)$ and $f(x)$ such that $f_n(x)$ converges to $f(x)$ pointwisely, $f_n^*(x)$ is convergent to $f^*(x)$ on $\beta(S)$.*

(3) *For every sequence of bounded continuous functions $f_n(x)$ such that $f_n(x)\downarrow 0$, we have $f_n^*(x)\rightarrow 0$ on $\beta(S)$.*

Proof. (1)$\rightarrow$(2) follows from Corollary 2 stated before. Since $S$ is dense in $\beta(S)$, (2) implies (3).

To prove (3)$\rightarrow$(1), it is sufficient to show that the Dini theorem on monotone sequence of bounded continuous functions holds true on $S$. I. Glicksberg [2], J. Mařik [6] and the present writer [3] have proved that the property is a characterisation of the pseudo-compactness. Let $f_n(x)$ be a sequence of bounded continuous functions such that $f_n(x)\downarrow 0$, then $f_1(x)\geq f_2(x)\geq\cdots$ implies $f_1^*(x)\geq f_2^*(x)\geq\cdots$ on $\beta(S)$, and by the condition (3), we have $f_n^*(x)\downarrow 0$. Since the Dini theorem holds true on $\beta(S)$, $f_n^*(x)$ is uniformly convergent to 0 on $\beta(S)$. Therefore $f_n(x)$ is uniformly convergent to 0 on $S$. This shows (3)$\rightarrow$(1).

## References

[1] R. G. Bartle: On compactness in functional analysis, Trans. Am. Math. Soc., **79**, 35-57 (1955).

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[4] K. Iséki: Pseudo-compactness and $\mu$-convergerce, Proc. Japan Acad., **33**, 368-371 (1957).