125. An Example of Kernel of Non-Carleman Type

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In this note, we construct an example of symmetric measurable kernel of non-Carleman type which determines a bounded self-adjoint operator in $L^2[0, 1]$[1] and has some additional properties stated in the following.

More precisely *we construct a function* $S(x, y)$ *on* $[0, 1]\times[0, 1]$ *with the following properties* (A), (B), (C), (D), (E), (F):

(A) $S(x, y)\geqq 0$, $S(x, y)=S(y, x)$ *on* $[0, 1]\times[0, 1]$.

(B) $S(x, y)$ *is a Baire's function of the 1st class on* $[0, 1]\times[0, 1]$.

(C) *If* $f(y)\in L^2[0, 1]$, $S(x, y)f(y)\in L^1[0\leqq y\leqq 1]$[2] *for every* $x\in[0, 1]-N_f$ *where* N_f *is a null set depending on* $f(y)$.

(D) $\int_0^1 S(x, y)f(y)dy \in L^2[0, 1]$ *if* $f(y)\in L^2[0, 1]$.

(E) *The operation* H *defined for all* $f(y)\in L^2[0, 1]$ *by*

$$H: f(y)\rightarrow \int_0^1 S(x, y)f(y)dy$$

is a bounded self-adjoint operator in $L^2[0, 1]$.

But

(F) $S(x, y)\notin L^2[0\leqq y\leqq 1]$[2] *for any* $x\in[0, 1]$.

§1. Kernel $K(x, y)$. We define three functions $R(n)$, $P(n)$, $Q(n)$ of integer $n\geqq 0$ by

$$R(0)=0,\ R(n)=\sum_{s=1}^{n} s^{-1} \quad \text{for } n\geqq 1$$

$$P(n)=R(n)-[R(n)]^{3)} \quad \text{for } n\geqq 0$$

$$Q(0)=0,\ Q(n)=6\pi^{-2}\sum_{s=1}^{n} s^{-2} \quad \text{for } n\geqq 1.$$

Then since $0<R(n)-R(n-1)\leqq 1$ for $n\geqq 1$, for $n\geqq 1$ $[R(n)]=[R(n-1)]$ or $[R(n)]=[R(n-1)]+1$ and if $[R(n)]=[R(n-1)]$, then $0\leqq P(n-1)<P(n)<1$ and if $[R(n)]=[R(n-1)]+1$, then $0\leqq P(n)\leqq P(n-1)<1$. Also it is well known that $Q(n)\rightarrow 1$ $(n\rightarrow\infty)$.

We define a function $K(x, y)$ on $[0, 1]\times[0, 1]$ in the following way.

For (x, y) such that $0\leqq x\leqq 1$ $Q(n-1)\leqq y<Q(n)$ $(n\geqq 1)$, we put

1) $M[0, 1]$, $L[0, 1]$, $L^2[0, 1]$ are the classes of bounded measurable, integrable, square integrable functions on the closed interval $[0, 1]$ respectively.

2) $f(x, y)\in L^2[0\leqq x\leqq 1]$ or $f(x, y)\in L^2[0\leqq y\leqq 1]$ means that $f(x, y)$ as a function of x or y belongs to $L^2[0, 1]$ for a particular value of y or x. Similarly for other function classes defined in 1).

3) $[a]$ is the greatest integer not greater than the real number a.