149. On Boundary Values of Some Pseudo-Analytic Functions

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Let $\zeta=\varphi(z)$ be a quasi-conformal mapping from $|z|<1$ to $|\zeta|<1$. Then $\varphi(z)$ is not necessarily absolutely continuous function of $\arg z=t$ on $|z|=1$ (cf. [4]), although it is always continuous and of bounded variation for $0\leq t\leq 2\pi$. In the present short note we shall give a sufficient condition, for $\varphi(e^{it})$ to be absolutely continuous in t, in such a form, that it is applicable to generalization of some classical theorems. Our analysis is based essentially on Ahlfors's mapping theory [2, 3]. We must also remark that the result is closely related to one of the propositions stated in [5] without proof.

In this paper we use the following notations: for any complex number z, z^* is its inversion with respect to the unit circumference. Areal mean of a continuous function $g(z)$ over the disk $|z-a|\leq b$ shall be denoted by $M(g;a;b)$, i.e.

$$M(g;a;b)=\frac{1}{\pi b^2}\int_0^b\int_0^{2\pi} g(a+re^{it})\,rdtdr.$$

Any integral without explicit indication of its integration domain should be computed over the whole plane.

Lemma. *Given any function $g(z)$ in $|z|<1$ which fulfils the Hölder condition of order α*

$$|g(z_1)-g(z_2)|\leq A|z_1-z_2|^{\alpha} \quad |z_1|<1,\ |z_2|<1,\ 0<\alpha\leq 1,$$

then there exists a sequence of functions $\{g_n(z)\}$ in $|z|<\infty$, such as to satisfy the conditions

i) *$g_n(z)$ has a uniformly bounded carrier,*

ii) $|g_n(z_1)-g_n(z_2)|\leq B|z_1-z_2|^{\alpha}$,

iii) $\sup_{|z|<\infty}|g_n(z)|\leq \sup_{|z|<1}|g(z)|$, $\quad\left(\begin{matrix}|z_1|<\infty,\ |z_2|<\infty\\ n=1,2,\cdots\end{matrix}\right)$

iv) *$\{g_n(z)\}$ converges uniformly to $g(z)$ in $|z|<1$.*

Proof. For example we proceed as follows:

Set

$$\gamma(z)=\begin{cases} g(z^*) & 1<|z|\leq 4\\ 0 & |z|\geq 5.\end{cases}$$

And in the circular ring $4<|z|<5$, $\gamma(z)$ shall be equal to the solution of Dirichlet problem with the boundary values $g(z^*)$ on $|z|=4$, 0 on $|z|=5$.

Let $\delta>0$ be a sufficiently small number, with which we define the function