145. On the Projection of Norm One in W^*-algebras

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In the present paper, we will study on the projection of norm one from any W^*-algebra onto its subalgebra. By a projection of norm one we mean a projection mapping from any Banach space onto its subspace whose norm is one. At first, we find some properties of a projection of norm one from a C^*-algebra to its C^*-subalgebra. These properties turn out to have some interesting applications to the recent theory of W^*-algebras, which we shall show in the following.

Through our discussions we denote the dual of a Banach space M and the second dual by M' and M'', respectively.

Theorem 1. Let M be a C^*-algebra with a unit and N its C^*-subalgebra. If π is a projection of norm one from M to N, then

1. *π is order preserving,* 2. *$\pi(axb)=a\pi(x)b$ for all $a, b \in N$,*
3. *$\pi(x)*\pi(x) \leq \pi(x*x)$ for all $x \in M$.*

Proof. Consider the second dual of M and N, M'' and N''. M'' is a W^*-algebra containing M as a σ-weakly dense C^*-subalgebra by Sherman's theorem (cf. [14, 15]), and N'' may be considered as a W^*-subalgebra of M'', for it is identified with the bipolar of N in M''. The second transpose of π, the extension of π to M'', is a projection of norm one from M'' to N''. Thus, it suffices to prove the theorem when M is a W^*-algebra and N a W^*-subalgebra of M. As in [5, Lemma 8] we can show that π is $*$-preserving and order preserving, which one can easily see since π is of norm one.

Next, take a projection e of N and $a \in M$, positive and $\|a\| \leq 1$. We have $e \geq eae$, whence $e \geq \pi(eae)$, so that $\pi(eae)=e\pi(eae)e$. Thus, we have $\pi(exe)=e\pi(exe)e$ for all $x \in M$. Take an element $x \in M$, $\|x\| \leq 1$. Put $\pi(ex(1-e))=x'$. Then

$$\|ex(1-e)+ne\| = \|\{ex(1-e)+ne\}\{(1-e)x*e+ne\}\|^{1/2}$$
$$= \|ex(1-e)x*e+n^2e\|^{1/2} \leq (1+n^2)^{1/2} \text{ for all integers } n.$$

On the other hand, if $\dfrac{ex'e+ex'^*e}{2} \neq 0$ we may suppose without loss of generality that this element has a positive spectrum $\lambda > 0$. Then,

$$\|x'+ne\| = \|ex'e+ne+ex'(1-e)+(1-e)x'e+(1-e)x'(1-e)\|$$
$$\geq \|e(x'+nl)e\| \geq \left\|\frac{ex'e+ex'^*e}{2}+ne\right\| \geq \lambda+n \text{ for all } n.$$

Therefore, $\|x'+ne\| \geq \lambda+n > (1+n^2)^{1/2} \geq \|ex(1-e)+ne\|$ for a sufficient-