In the present paper, we will study on the projection of norm one from any $W^*$-algebra onto its subalgebra. By a projection of norm one we mean a projection mapping from any Banach space onto its subspace whose norm is one. At first, we find some properties of a projection of norm one from a $C^*$-algebra to its $C^*$-subalgebra. These properties turn out to have some interesting applications to the recent theory of $W^*$-algebras, which we shall show in the following.

Through our discussions we denote the dual of a Banach space $M$ and the second dual by $M'$ and $M''$, respectively.

**Theorem 1.** Let $M$ be a $C^*$-algebra with a unit and $N$ its $C^*$-subalgebra. If $r$ is a projection of norm one from $M$ to $N$, then
1. $r$ is order preserving,
2. $r(axb) = aπ(x)b$ for all $a, b ∈ N$,
3. $r(x) ⋅ r(x) ≤ π(x ⋅ x)$ for all $x ∈ M$.

**Proof.** Consider the second dual of $M$ and $N$, $M''$ and $N''$. $M''$ is a $W^*$-algebra containing $M$ as a $σ$-weakly dense $C^*$-subalgebra by Sherman's theorem (cf. [14, 15]), and $N''$ may be considered as a $W^*$-subalgebra of $M''$, for it is identified with the bipolar of $N$ in $M''$. The second transpose of $π$, the extension of $π$ to $M''$, is a projection of norm one from $M''$ to $N''$. Thus, it suffices to prove the theorem when $M$ is a $W^*$-algebra and $N$ a $W^*$-subalgebra of $M$. As in [5, Lemma 8] we can show that $π$ is $*$-preserving and order preserving, which one can easily see since $π$ is of norm one.

Next, take a projection $e$ of $N$ and $a ∈ M$, positive and $∥a∥ ≤ 1$. We have $e ≥ eae$, whence $e ≥ π(eae)$, so that $π(eae) = π(eae)e$. Thus, we have $π(exe) = eπ(exe)e$ for all $x ∈ M$. Take an element $x ∈ M$, $∥x∥ ≤ 1$. Put $π(ex(1−e)) = x'$. Then

$$∥ex(1−e)+ne∥ ≤ ∥ex(1−e)−ne∥(1−e)x' + ne(1−e)x'∥^{1/2} ≤ (1+n^2)^{1/2}$$

for all integers $n$.

On the other hand, if $ex' + ex'^*e = 0$ we may suppose without loss of generality that this element has a positive spectrum $λ > 0$. Then,

$$∥x' + ne∥ ≥ ∥ex' + ne + ex'(1−e) + (1−e)x' + (1−e)x'(1−e)∥ ≥ ∥e(x' + nl)e∥ ≥ ∥ex' + ex'^*e + ne∥ ≥ λ + n$$

for all $n$. Therefore, $∥x' + ne∥ ≥ λ + n > (1+n^2)^{1/2} ≥ ∥ex(1−e) + ne∥$ for a sufficient-