144. On a Theorem on Function Space of A. Grothendieck

By Kiyoshi ISÉKI

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Very interesting results on countable compactness of function spaces have been obtained by A. Grothendieck [3]. In this paper, we shall consider the case of a set of real valued continuous functions on a pseudo-compact space, and we shall give a generalisation of his result. Let E be a pseudo-compact space, and let $C_s(E)$ be *the topological space of all continuous functions on* E *with simple convergence topology.* Then *if a subset* A *of* $C_s(E)$ *is conditionally compact, then it is conditionally countably compact.* Next, *if* A *is conditionally countably compact, for every sequence* $\{f_m\}$ *of* A *and countable set* $\{x_m\}$ *of* E, *there is a continuous function* $f(x)$ *on the closure* C *of* $\{x_n\}$ *such that, for every* x *of* C, $f(x)$ *is a cluster point of* $f_n(x)$ *and* $\{f_n(x)\}$ *is pointwise bounded.*

Let $\{f_m\}$ be a countable set of $C_s(E)$, and let $\{x_n\}$ be a countable set of E. Then following A. Grothendieck [3] we shall define *a double cluster point* α *of the double sequence* $\{f_m(x_n)\}$ as follows.

A point (number) α is said to be a *double cluster point* of $\{f_m(x_n)\}$, if, for each neighbourhood U of α, and a given integer N, there are infinitely many $f_m(x_n)$ meeting U for $m, n\geq N$.

If $\{f_m\}$ *and* $\{x_n\}$ *satisfy the conditions in the previous section, then* $\{f_m(x_n)\}$ *has at least one double cluster point.* To prove it, we shall define an equivalent relation on E. For x, y of E, we define $\rho(x, y)$ by

$$\rho(x, y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|f_n(x)-f_n(y)|}{1+|f_n(x)-f_n(y)|}$$

(for example, see R. G. Bartle [1, p. 48]).

By $\rho(x,y)=0$ we shall define an equivalent relation $x\sim y$. Then the space E is decomposed into equivalent classes by the relation "$\sim$". By $\mathfrak{E}$, we denote the set of equivalent classes. Then $\mathfrak{E}$ is a metric space with the metric ρ. By the continuity of the natural mapping $F:E\to\mathfrak{E}$, $\mathfrak{E}$ is a compact metric space with respect to ρ and $X^\rho=F(x)$ is the class containing x. For $f\in C_s(E)$, we shall define φ on $\mathfrak{E}$ by $\varphi(X^\rho)=f(x)$, where $X^\rho=F(x)$. Therefore, for f_m and f, we have continuous functions $\varphi_n(X^\rho)$, $\varphi(X^\rho)$ on X^ρ for $x\in C$. Since $\mathfrak{E}$ *is compact*,[1] the set X^ρ for x of C has at least one cluster point X_0^ρ. Therefore let α be $\varphi(X_0^\rho)$, then we obtain that α is a double cluster of $f_m(x_n)$. It is sufficient to show that $\varphi(X_0^\rho)$ is a double cluster point of $\varphi_m(X_n^\rho)$, where $X_n^\rho=F(x_n)$. Since X_0^ρ is a cluster point of $\{X_n^\rho\}$, there is a subsequence $\{X_{n_i}^\rho\}$ which

1) See. K. Iséki [5, p. 424].