141. *On Sobolev-Friedrichs' Generalisation of Derivatives*

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We consider a fixed domain (open set) G in $R^n(x_1,\cdots,x_n)$ and in this note a function means always a complex valued measurable function defined on G and s is always a fixed non-negative integer. We identify two functions which coincide except on a null set. We use the following notations and definitions:

θ: the void set.

T_s: the set of finite sequences $(i_1, i_2,\cdots, i_p)$ of integers such that $1\leqq i_1, i_2,\cdots, i_p\leqq n$ $0\leqq p\leqq s$. The only sequence $(i_1,\cdots, i_p)$ for $p=0$ is the void set θ by definition.

D_i: the differentiation with respect to the variable x_i.

$$D_{(i_1,i_2,\cdots,i_p)}=D_{i_1}\cdot D_{i_2}\cdots D_{i_p} \quad (p\geqq 1).$$

$$D_\theta=I \text{ (the identity operator).}$$

$(f, g)^A=\int_A f\cdot\bar{g}dx_1\cdots dx_n$ for two functions f, g if $f\cdot\bar{g}$ is (Lebesgue) integrable on a measurable subset A of G.

$$||f||^A=\left(\int_A |f|^2 dx_1\cdots dx_n\right)^{1/2}.$$

We write (f, g), $||f||$ for $(f, g)^G$, $||f||^G$ respectively.

H: the set of functions such that $||f||<+\infty$.

$\mathfrak{H}$: the set of functions such that $||f||^A<+\infty$ for every compact set A contained in G.

C_s: the set of functions having continuous partial derivatives up to order s on G.

C_∞: the set of functions infinitely continuously differentiable on G.

Further following the authors above cited, we state some definitions and some propositions related to them for whose proofs we refer to K. O. Friedrichs [1, 2], Sobolev [6], L. Nirenberg [5]. If for a function U on G there are a set of functions $U_{(i_1,\cdots,i_p)}\varepsilon\mathfrak{H}$ $((i_1,\cdots, i_p)\varepsilon T_s)$ and a sequence of functions $f_m\varepsilon C_s$ $(m=1, 2,\cdots)$ such that $U_\theta=U$ and $||D_{(i_1,\cdots,i_p)}f_m-U_{(i_1,\cdots,i_p)}||^A\to 0$ $(m\to\infty)$ for every compact set A contained in G and for every $(i_1,\cdots, i_p)\varepsilon T_s$, then U is said *strongly differentiable up to order s on G* and $U_{(i_1,\cdots,i_p)}$ are said *strong derivatives of U of order p*. We denote the set of functions strongly differentiable up to order s on G by $\mathfrak{H}_s$.

The strong derivative $U_{(i_1,\cdots,i_p)}$ of $U\varepsilon\mathfrak{H}_s$ is uniquely determined for each $(i_1,\cdots, i_p)\varepsilon T_s$.

$C_s\subset\mathfrak{H}_s$ and the strong derivative $U_{(i_1,\cdots,i_p)}$ of $U\varepsilon C_s$ is equal to