5. *Two Theorems on Fourier Transform*

By Takao KAKITA

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In "Théorie des Distributions" L. Schwartz stated without complete proof that two topological vector spaces $\mathcal{O}_M$ and $\mathcal{O}'_C$ are topologically isomorphic by Fourier transform. Here we shall give a full proof of the theorem.

On the other hand, according to K. Nomizu, a theorem of the same type was proved in C. Chevalley's lecture. The only difference is that Chevalley introduced into $\mathcal{O}'_C$ the bounded-open topology, regarding it as the space $L(\mathfrak{S}, \mathfrak{S})$ of continuous linear operators from $\mathfrak{S}$ to $\mathfrak{S}$. Thus the proof which we shall give concludes that Schwartz's topology in $\mathcal{O}'_C$ and Chevalley's topology coincide.

Next, in 1936, M. Plancherel and G. Polya have proved two main theorems in their paper [1]; one is an extension of the Paley-Wiener's theorem to the multiple integrals of Fourier, and the other a theorem on a simple relation between the spectre of an entire function of exponential type and the order of increase of the function in different directions. The former has been generalized by L. Schwartz [2] to the case with which distributions with compact carriers are concerned.

Similar generalization of the latter theorem of Plancherel-Polya was indicated by Schwartz. But no proof seems to have been published. In this paper, we shall give the formulation and the proof of the theorem.

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1. Schwartz's theorem. By $\mathfrak{S}^0$ we mean the space of rapidly decreasing continuous functions with the topology defined by semi-norms

$$\rho_k(f)=\sup_x (1+r^2)^k \,|f(x)| \qquad (f\in\mathfrak{S}^0)$$

where k is a positive integer.

We shall give two lemmas.

Lemma 1.1. *Let B be a bounded set of $\mathfrak{S}^0$. Then there exist a function $\varphi\in\mathfrak{S}$ and a constant c such that $|f(x)|\leqq c\,|\varphi(x)|$ for all $f\in B$.*

The proof is classical and so omitted.

Lemma 1.2. *Let $P(x)$ be a polynomial on R^n. Then PT belongs to $\mathcal{O}'_C$ when $T\in\mathcal{O}'_C$, and the mapping $T\in\mathcal{O}'_C\to PT\in\mathcal{O}'_C$ is continuous, according to Schwartz's topology.*

Proof. Let the degree of $P(x)$ be m. It is obvious that $P\varphi(1+r^2)^{-m}\in\mathcal{D}_{L^1}$ for each $\varphi\in\mathcal{D}_{L^1}$. So we have $PT\in\mathcal{O}'_C$, since

$$(1+r^2)^k PT\cdot\varphi=(1+r^2)^{k+m}T\cdot P\varphi(1+r^2)^{-m}.$$