2. Duality in Mathematical Structure

By Tadashi OHKUMA
Department of Mathematics, Tokyo Institute of Technology, Tokyo
(Comm. by Z. SUETUNA, M.J.A., Jan. 13, 1958)

1. In many representation theorems of different branches of abstract mathematics, for example, representation of ordered sets by cuts, that of lattices by sets of ideals, the conjugate spaces of Banach spaces, or the duality of groups, there seem to appear some similar conceptions. In order to deal with those representation theorems simultaneously, we made an attempt to set up a concept of a universal mathematical structure, in which the main rôle is played by some selected applications of a system to a system, which we call homomorphisms, but they may be continuous mappings in topological spaces, order-preserving mappings in ordered sets or linear operators in linear spaces.

The definitions of our structure and some results of them will be stated in this note, but we omit the detail of proofs which, as well as the applications to individual specialized systems, will be published elsewhere.

2. Let $\mathfrak{S}$ be a family of sets. A set in $\mathfrak{S}$ is called a *system*. If X and Z are systems and $Z \subset X$, then Z is called a *subsystem* of X. We assume that, to each pair of systems X and Y, a family $\mathrm{Hom}(X, Y)$ of applications which map X into Y is distinguished. An application φ in $\mathrm{Hom}(X, Y)$ is called a *homomorphism* of X in Y. Further we assume that those homomorphisms and the family $\mathfrak{S}$ satisfy the following axioms which fall into five groups (A)-(E).

In these statements of axioms, the letters X, Y and Z denote systems.

The axioms of the group (A) are concerned with the conditions for an application of a system to be a homomorphism.

(A1) *If $Z \subset X$, and I_Z is the identical mapping on Z, then $I_Z \in Hom(Z, X)$.*

(A2) *If $\varphi \in Hom(X, Z)$ and $\psi \in Hom(Z, Y)$, then $\psi\varphi \in Hom(X, Y)$.*

(A3) *If $\varphi \in Hom(X, Y)$ and $\varphi(X) \subset Z$, then $\varphi \in Hom(X, Z)$.*

The axioms of the group (B) are concerned with the conditions for a set to be a system.

(B1) *If $\varphi \in Hom(X, Y)$, then $\varphi(X) \in \mathfrak{S}$.*

(B2) *If $\varphi \in Hom(X, Y)$ and $Z \frown \varphi(X) \neq \phi$, then $\varphi^{-1}(Z) \in \mathfrak{S}$.*[1]

(B3) *There exists a one element system $\{e\}$ such that for any system X the application which maps each element x of X onto e is*

1) $\varphi(X) = \{\varphi(x);\ x \in X\}$, $\varphi^{-1}(Z) = \{x;\ \varphi(x) \in Z\}$.