1. On Zeta-Functions and L-Series of Algebraic Varieties

By Makoto ISHIDA

Mathematical Institute, University of Tokyo

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In this paper, we shall prove Weil's conjecture on zeta-functions for algebraic varieties, defined over finite fields, having abelian varieties as abelian (not necessarily unramified) coverings and also Lang's analogous conjecture on L-series for those coverings. Then we shall see some interesting relation between the zeta-functions of such algebraic varieties and those of their Albanese varieties. Moreover those results will enable us to prove Hasse's conjecture on zeta-functions for some algebraic varieties defined over algebraic number fields. In the following we shall use the definitions, notations and results of Weil's book [6] often without references.

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1. Let V be a normal projective variety of dimension r, defined over a finite field k with q elements; let A be an abelian variety such that $f: A\rightarrow V$ is a Galois (not necessarily unramified) covering, also defined over k, with group G and of degree n (cf. Lang [2]). The map $a\rightarrow a^q$ for all points a on A determines an endomorphism of A, which is denoted by $\pi=\pi_A$. Let x be a generic point of A over k. Then, for σ in G, the map $x\rightarrow x^\sigma$ induces a birational transformation of A defined over k; hence we can write $x^\sigma=\eta_\sigma(x)+a_\sigma$ where η_σ is an automorphism of A defined over k and a_σ is a rational point on A over k.

Now we consider an endomorphism $\pi^m-\eta_\sigma$ of A for a positive rational integer m and for σ in G. As $k(\eta_\sigma(x))=k(x)$, we have $k(x^{q^m}, (\pi^m-\eta_\sigma)(x))=k(x)$ and so $\nu_i(\pi^m-\eta_\sigma)=1$. Hence the order of the kernel of this endomorphism is equal to $\det M_l(\pi^m-\eta_\sigma)$, with a rational prime l different from the characteristic of k, which is denoted by $\nu(m,\sigma)$. As $\det M_l(\eta_\sigma)=1$ and the matrix $M_l(\pi^m\eta_\sigma^{-1}-1)$ is of even degree $2r$, we have also $\nu(m,\sigma)=\det M_l(1-\pi^m\eta_\sigma^{-1})$.

Then the L-series $L(u,\chi,A/V)$ of the covering A/V belonging to an irreducible character χ of G is given by the following logarithmic derivative:

$$d/du\cdot\log L(u,\chi,A/V)=\textstyle\sum_{m=1}^{\infty}\{1/n\cdot\sum_{\sigma\in G}\chi(\sigma)\nu(m,\sigma)\}u^{m-1}$$

Theorem 1. *Let $Z(u, V)$ and $Z(u, A)$ be the zeta-functions of V and A over k. Then we have the equality $Z(u, V)=Z(u, A)$ if and*