21. Some Properties of $(n-1)$-Manifolds in n-Space

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(Comm. by K. KUNUGI, M.J.A., Feb. 12, 1958)

In this note we shall give a brief account of some properties of a polyhedral $(n-1)$-manifold in the n-dimensional Euclidean space R^n, that is, of a triangulable $(n-1)$-manifold P^{n-1} rectilinearly imbedded in R^n. Theorems 1, 2, 3, 4 relate to the differentiable approximations of P^{n-1} in R^n and Theorems 5, 6 relate to the curvatura integra of P^{n-1} in R^n. Full details will appear in Osaka Mathematical Journal.

1. Let S be a point set in some Euclidean space R^n. A k-plane H^k $(k\geqq 1)$ in R^n will be called *transversal to S* if there exists a positive number ε such that a line through any two points of S makes an angle greater than ε with H^k. A k-plane $H^k(p)$ through a point p of S will be called *transversal to S at p* if $H^k(p)$ is transversal to some neighbourhood of p on S.

Let M^m be a topological manifold (with or without boundary) in some Euclidean space R^n. We shall say that M^m is *in normal position* in R^n if it is possible to define through each point p of M^m an $(n-m)$-plane $H^{n-m}(p)$ which varies continuously with p and is transversal to M^m at p. Let P^m be a polyhedral m-manifold in R^n. Then we shall say that P^m is *in locally normal position* in R^n if the star of any vertex on P^m is in normal position in R^n. Then we obtain the following:

Theorem 1. *Any polyhedral $(n-1)$-manifold P^{n-1} in locally normal position in the n-dimensional Euclidean space R^n is in normal position.*

Outline of the proof: Let ε be a positive number less than $\frac{1}{n}$. Let s^j be any j-simplex of P^{n-1} and let s^{n-1} be any $(n-1)$-simplex of P^{n-1} which belongs to the star of s^j on P^{n-1}. We choose barycentric coordinates $(u_0, u_1,\cdots, u_{n-1})$ on s^{n-1} so that $u_{j+1}=\cdots=u_{n-1}=0$ at s^j. Let $N_{s^{n-1}}(s^j)$ be the set of points whose barycentric coordinates $(u_0,\cdots, u_{n-1})$ satisfy the following:

$$\varepsilon\leqq u_0,\cdots, \varepsilon\leqq u_j,\ 0\leqq u_{j+1}\leqq\varepsilon,\cdots,\ 0\leqq u_{n-1}\leqq\varepsilon.$$

We shall define

$$N(s^j)=\sum_{s^{n-1}\in St(s^j)} N_{s^{n-1}}(s^j)$$

where $St(s^j)$ is the star of s^j on P^{n-1}.

Thus P^{n-1} is covered by these closed $(n-1)$-dimensional regions $N(s^j)$ which are disjoint from each other except eventually for common