20. On Symmetric Skew Unions of Knots

By Yoko HASHIZUME*) and Fujitsugu HOSOKAWA**)

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Introduction. S. Kinoshita and H. Terasaka introduced the notion of *symmetric unions and symmetric skew unions* of knots and showed that the Alexander polynomial of the symmetric union of a knot is the square of that of the original knot. As regards the symmetric skew union of a knot nothing more is obtained than that its Alexander polynomial $\Delta(x)$ is independent of the winding number. In this note we shall give a more explicit form of $\Delta(x)$ and show especially that this is of the form $\phi(x)\cdot\phi(1/x)$.[1]

1. We shall call a polynomial $f(x)$ *symmetric* (*skew symmetric*) if $f(x)=x^p f(1/x)$ $(f(x)=-x^p f(1/x))$ for a suitable integer p. We shall call the integer $n-m$ the *reduced degree* of a polynomial $f(x)=a_l x^l+\cdots+a_n x^n+\cdots+a_m x^m+\cdots+a_1 x+a_0$ if $a_l=a_{l-1}=\cdots=a_{n+1}=0$, $a_n\neq 0$, $a_m\neq 0$ $(n>m)$ and $a_{m-1}=\cdots=a_1=a_0=0$.

Lemma 1. *Let $f(x)$ and $F(x)$ be symmetric polynomials with even reduced degrees and let $g(x)$ and $G(x)$ be skew symmetric polynomials, such that*

$$F(x)=xf(x)+(x-1)g(x)$$
$$G(x)=(1-x)f(x)+g(x).$$

Then, if

$$f(x)=a_n x^n+\cdots+a_m x^m$$
$$g(x)=b_n x^n+\cdots+b_m x^m$$

where $n>m$, and a_n or $b_n\neq 0$ and a_m or $b_m\neq 0$, we have either

$$\text{(I)}\qquad \begin{cases} f(x)=a_n x^n+\cdots+a_{m+1}x^{m+1}+a_m x^m \\ g(x)=b_n x^n+\cdots+b_{m+1}x^{m+1} \end{cases}$$

where $a_n=a_m\neq 0$ and $b_{n-i}=-b_{(m+1)+i}$ $(i=1,2,\cdots,n-(m+1))$, or

$$\text{(II)}\qquad \begin{cases} f(x)=\phantom{b_n x^n+{}} a_{n-1}x^{n-1}+\cdots+a_m x^m \\ g(x)=b_n x^n+b_{n-1}x^{n-1}+\cdots+b_m x^m \end{cases}$$

where $b_n=-b_m\neq 0$ and $a_{(n-1)-i}=a_{m+i}$ $(i=1,2,\cdots,n-(m+1))$.

Proof. By the conditions

$$F(x)=(a_n+b_n)x^{n+1}+(a_{n-1}+b_{n-1}-b_n)x^n+\cdots+(a_m+b_m-b_{m+1})x^{m+1}-b_m x^m$$
$$G(x)=-a_n x^{n+1}+(a_n+b_n-a_{n-1})x^n+\cdots+(a_{m+1}+b_{m+1}-a_m)x^{m+1}+(a_m+b_m)x^m.$$

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*) Department of Mathematics, Osaka University.
**) Department of Mathematics, Kobe University.

1) This ascertains the result of R. H. Fox and J. W. Milnor [2], for any symmetric (skew) unions of knots may easily be proved to belong to the category of knots considered by them.