19. On Strictly Continuous Convergence of Continuous Functions

By Takesi ISIWATA
Tokyo University of Education, Tokyo

(Comm. by K. KUNUGI, M.J.A., Feb. 12, 1958)

1. Let X be a topological space and $C(X)$ be the set of all real-valued continuous functions defined on X. A topology of $C(X)$ is said to be *admissible* provided that $f(x)$ is jointly continuous with respect to the given topologies of X and $C(X)$ respectively. We denote by "$\{f_n\}\to f$ (*jointly*)" that a sequence $\{f_n\}$ converges to f with respect to some admissible topology of $C(X)$. A sequence $\{f_n\}$ is said to be *continuously convergent* to f (abbreviated to $\{f_n\}\to f$ (*cont.*)) if $\{x_n\}\to x$ implies $\{f_n(x_n)\}\to f(x)$. A sequence $\{f_n\}$ is said to be *strictly continuous convergent* to f (abbreviated to $\{f_n\}\to f$ (*str. cont.*) if $\{f(x_n)\}\to\alpha$, then $\{f_n(x_n)\}\to\alpha$ where α is a real number. Finally we shall define "$\{f_n\}\to f$ (*uniformly*)" when a sequence $\{f_n\}$ is uniformly convergent to f. For simplicity, by the property (S), we shall mean the following:

(S): $\{f_n\}\to f$ (*cont.*) *implies* $\{f_n\}\to f$ (*str. cont.*).

Recently, Iséki [1-3] investigated the relations between concepts of (strictly) continuous convergence, pseudo-compactness and countable compactness. In this paper, we shall prove the following:

Theorem 1. *Let X be a countably compact T_1-space. Then $\{f_n\}\to f$ (jointly) if and only if $\{f_n\}\to f$ (str. cont.)* (hence by Theorem 2 in [1], $\{f_n\}\to f$ *(jointly) if and only if* $\{f_n\}\to f$ *(uniformly)*).

Theorem 2. *Let Z be any topological space and X be any dense subset of Z. If X has the property (S), then Z has the property (S).*

The converse of Theorem 2 is not necessarily true (cf. Example 1 below).

Corollary. *Let X be a completely regular T_1-space, and Z be the Čech compactification of X. If X has the property (S), then any subspace Y of Z, $X\subset Y$, has the property (S).*

From Corollary, we can construct a pseudo-compact space which has the property (S) without being countably compact (cf. Example 2 below). Finally, we shall show the existence of a compact space which has not the property (S), by the following

Theorem 3. *Let X be any discrete space containing infinitely many points, and Z be the Čech compactification of X; then we have the following statements*:

i) *Z has no convergent sequence.*