32. *Measures in the Ranked Spaces*

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This paper is an attempt to introduce the notion of measure in ranked spaces[1] which is an extension of the measure in the sense of Lebesgue.[2] This will be the first step to the general measure theory in the ranked spaces.

In Section 1, as the preparation for Section 2, we study some properties of outer measures in topological spaces. In Section 2 we give outer measures in ranked spaces and study their properties. Some examples will be given in Section 3.

1. Let R be a space whose topology is given by a system of neighbourhoods which satisfies F. Hausdorff's axioms (A), (B) and (C).[3]

Definition 1.[4] A set function Γ, defined on the family of all subsets of R, is called an outer measure in R if the following conditions (1.1)-(1.4) are satisfied:

(1.1) $0 \leq \Gamma(A) \leq +\infty$ for any subset A of R.

(1.2) $\Gamma(0)$[5]$=0$.

(1.3) $\Gamma(A) \leq \Gamma(B)$ whenever $A \subseteq B$.

(1.4) $\Gamma(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \Gamma(A_n)$ for every countable sequence $\{A_n\}$ of subsets of R.

Definition 2.[4] Let Γ be an outer measure in R. A subset A of R is Γ-measurable if, for every subset E of R, we have

(1.5) $$\Gamma(E) = \Gamma(E \frown A) + \Gamma(E \frown (R - A)).$$[6]

Theorem 1. *Let Γ be an outer measure in R which satisfies the following conditions* (1.6)-(1.9):

(1.6) *For every disjoint finite or countable family $\{v_n(p_n);\ n=1, 2, \cdots\}$ of neighbourhoods, $\Gamma(\overline{\bigcup_n v_n(p_n)}$[7]$-\bigcup_n v_n(p_n))=0$ if $\Gamma(\bigcup_n v_n(p_n)) < +\infty$.*

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1) The notion of ranked spaces was introduced by Prof. K. Kunugi in the notes "Sur les espaces complets et régulièrement complets. I-III", Proc. Japan Acad., **30**, 553-556, 912-916 (1954); **31**, 49-53 (1955).

2) F. Hausdorff and A. Appert studied this problem in the metric spaces. S. Saks: Theory of the Integral (1937). A. Appert: Mesures normales dans les espaces distanciés, Bull. Sci. Math., **60**, 329-352, 368-380 (1936).

3) F. Hausdorff: Grundzüge der Mengenlehre, 213 (1914).

4) Cf. P. Halmos: Measure Theory (1954).

5) 0 denotes the empty set.

6) For two subsets A and B of R, $A-B$ denotes the set of all elements p such that $p \in A$ and $p \notin B$.

7) For any subset A of R, $\bar{A}$ denotes the closure of A.