30. On a Generalization of the Concept of Functions

By Mikio SATO

(Comm. by Z. SUETUNA, M.J.A., March 12, 1958)

1. L. Schwartz has generalized the concept of functions on a C^∞-manifold by introducing his notion of *distributions*, which revealed to be most useful in various branches of analysis.[1] We now propose to introduce another generalization of the function concept in case the underlying manifold is C^ω (instead of C^∞) in utilizing the boundary values of analytic functions. This new concept will comprise that of Schwartz's distributions in case of C^ω-manifold. We shall call a *hyperfunction* a "function" in this generalized sense, defined precisely as follows. (For brevity, we give here the definition of the hyerfunction only in the 1-dimensional case, though we can define it for n-dimensional C^ω-manifolds.)[2]

Let R be the real axis $(-\infty, \infty)$ which we shall consider as lying in the complex plane $\boldsymbol{C}$. Let N be a locally compact subset of R. The family of all the "complex nbds of N", i.e. the open sets $D, D_1, D_2, \cdots$ of $\boldsymbol{C}$ which contain N as a closed subset, will be denoted by $\mathfrak{D}(N)$. On the other hand we shall denote, for any open set G of $\boldsymbol{C}$, the set of analytic (i.e. single valued regular analytic) functions in G with $\mathfrak{A}(G)$. $\mathfrak{A}(G)$ forms a ring, and if $D_1 \supset D_2$, $D_i \in \mathfrak{D}(N)$, we have clearly natural homomorphisms of $\mathfrak{A}(D_1)$ in $\mathfrak{A}(D_2)$ and of $\mathfrak{A}(D_1-N)$ in $\mathfrak{A}(D_2-N)$. The inductive limit of rings $\mathfrak{A}(D-N)$, $D \in \mathfrak{D}(N)$, will be denoted with $\tilde{\mathcal{A}}_N$, and that of $\mathfrak{A}(D)$, $D \in \mathfrak{D}(N)$, with $\mathcal{A}_N$. $\tilde{\mathcal{A}}_N$ is then considered as an extension ring of $\mathcal{A}_N$.

The quotient $\mathcal{A}_N$-module of $\tilde{\mathcal{A}}_N$ mod. $\mathcal{A}_N$ will be denoted by $\mathcal{B}_N$, and the elements of $\mathcal{B}_N$ generally by $g(x)$. These elements will be called hyperfunctions (h. f.) on N. A h. f. $g(x)$ is given by a function $\varphi(z) \in \mathfrak{A}(D-N)$ for some $D \in \mathfrak{D}(N)$. $\varphi(z)$ is called a *defining function* of $g(x)$, and we shall write

$$(1) \qquad g(x)=\varphi(x)|^{+}-\varphi(x)|^{-} \quad \text{or} \quad g(x)=\varphi(x+i0)-\varphi(x-i0).$$

It is easy to show that for every $g(x) \in \mathcal{B}_N$, there exists an open set $M \subset R$, such that $g(x)$ can be regarded as an element of $\mathcal{B}_M$,

1) L. Schwartz: Théorie des Distributions, I, II, Paris, Hermann (1950-1951).

2) After I had completed the manuscript of this note, I was kindly informed by Professor A. Weil through Professor Iyanaga that the same notion as that of "hyperfunction" had been already introduced by Professor G. Köthe in his paper: Die Randverteilungen analytischer Funktionen, Math. Z., **57** (1952). The content of §§ 2, 4 of the present note is also essentially contained in the paper of Professor Köthe, but the localization theorem and the extension to the case of n-dimensional manifolds are not considered in that paper.