49. On the Recurrence Theorems in Ergodic Theory

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1. For an ergodic, measure-preserving, one-to-one point transformation on a space of finite measure, M. Kac [2] made an interesting recurrence theorem which evaluates the value of the integral of a recurrence time. In this note we shall first state a recurrence theorem (Theorem 1) which enlightens the asymptotic behavior of a recurrence time. Next, on using the theorem, we shall give another proof of the Kac theorem (Theorem 2).

2. Let $(X, \mathfrak{F}, \mu)$ be a measure space such that X is an abstract space, $\mathfrak{F}$ a σ-field of subsets of X and μ a finite measure on $\mathfrak{F}$. It is supposed that $X \in \mathfrak{F}$. Let T be a measure-preserving, single-valued (not necessarily one-to-one) point transformation of X into itself, that is,

$$T^{-1}E=\{x; Tx \in E\} \in \mathfrak{F} \quad \text{and} \quad \mu(T^{-1}E)=\mu(E)$$

for any $E \in \mathfrak{F}$.

Before stating the definition of a recurrence time we recall a well-known

Recurrence theorem. *For every set $E \in \mathfrak{F}$ we can choose a set $N \in \mathfrak{F}$ of measure zero such that for each $x \in E-N$ there exists a positive integer $n(x)$ which satisfies $T^{n(x)}x \in E$* (for example, see [1], p. 10).

Let E be a set in $\mathfrak{F}$. The *recurrence time* $r(x)=r(x, E)$ denotes, for each $x \in E$, the least positive integer such that $T^{r(x)}x \in E$. Then $r(x)$ is defined almost everywhere in E by virtue of the recurrence theorem stated above. Further we define $r(x)=0$ for each $x \notin E$.

Theorem 1. *For every $E \in \mathfrak{F}$, the recurrence time $r(x)$ is an integrable function and*

$$(1) \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} r(T^j x) \begin{cases} =1 & \textit{for almost all } x \in E, \\ \leqq 1 & \textit{for almost all } x \notin E. \end{cases}$$

Theorem 2. *For every $E \in \mathfrak{F}$,*

$$(2) \qquad \mu(E) \leqq \int_E r(x)\mu(dx) \leqq \mu(X).$$

Moreover, T is ergodic if and only if

$$(3) \qquad \int_E r(x)\mu(dx)=\mu(X)$$

for every $E \in \mathfrak{F}$ of positive measure.