44. *Decomposition-equivalence and the Existence of Non-measurable Sets in a Locally Compact Group*

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Let G be a locally compact and σ-compact group and m^* a left invariant outer measure in G. In the theory of Haar's measure, it is well known that any two measurable sets $A\subseteq G$ and $B\subseteq G$ of the same measure are decomposition-equivalent to each other, that is, there exist direct decompositions

$$(1)\qquad \begin{aligned} A&=M+A_1+A_2+\cdots+A_n+\cdots,\\ B&=N+B_1+B_2+\cdots+B_n+\cdots \end{aligned}$$

of A and B, with relations

$$(2)\qquad m^*(M)=m^*(N)=0,\ \ g_iA_i=B_i,\ \ g_i\in G,\ \ i=1,2,\cdots,$$

and

(3) every A_i is m^*-measurable.

Conversely, any two measurable sets which are decomposition-equivalent to each other have clearly the same measure. Hence if $m(A)\neq m(B)$ (for measurable set A we write $m(A)$ instead of $m^*(A)$), then the set A is not decomposition-equivalent to B. But if we admit, in the expression (1), non-measurable sets A_i's, then it is proved that for any two measurable sets A and B of positive measures, even though $m(A)$ is not equal to $m(B)$, there exist direct decompositions (1) satisfying the condition (2). This is included in the Corollary of Theorem 1 as a special case.

Definition. Let A and B be two subsets of G. If there exist direct decompositions (1) satisfying the condition (2), then A is called to be almost decomposition-equivalent to B. And if further in the expression (1) both M and N can be taken to be empty, A is called to be completely decomposition-equivalent to B.

Remark 1. In the above definition it is not assumed that each A_i is measurable. Our definition of decomposition-equivalence is different from the usual one.

Notation. In the following we denote by $A\sim B$ and $A\approx B$ the almost and completely decomposition-equivalence of A to B respectively.

Remark 2. Suppose that $A\approx B$. Then $m^*(A)=0$ implies $m^*(B)=0$. And if A is of the first category, then B is also of the same category.

The following lemma is easily proved.

Lemma 1. 1) $A\sim A\,(A\approx A)$, 2) $A\sim B\,(A\approx B)$ implies $B\sim A\,(B\approx A)$, 3) $A\sim B$ $(A\approx B)$ and $B\sim D$ $(B\approx D)$ imply $A\sim D$ $(A\approx D)$.

Lemma 2. Suppose that $A\approx B$ $(A\sim B)$ and $B\subseteq A$. If $B\subseteq D\subseteq A$,