56. On Homomorphic Mappings

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In the theory of real valued functions we have

"Theorem A. Let R be the space of real numbers and $f(x)$ an additive function defined on R. If $f(x)$ is measurable (with respect to the Lebesgue measure), then $f(x)$ is continuous".

This is a well-known theorem. It will be natural to propose the following question, in connection with the above theorem:

"Let G and G^* be two topological groups and $\varphi(x)$ a homomorphic mapping of an abstract group G into an abstract group G^*. Under what conditions does it follow that $\varphi(x)$ is a continuous mapping of the topological group G into the topological group G^*"?

It is the purpose of the present paper to answer this question. First we shall extend Theorem A to a more general case (see Theorem 1). This generalization is the first answer for the above question. Next we shall prove a theorem (Theorem 2) which is the second answer for the above question. And we have, using our Theorems 1 and 2 and the duality theorem of Pontrjagin, an interesting consequence (see Theorem 3).

Definition 1. Let G be an abstract space and m^* an outer measure in G. Let f be a mapping of G into a topological space Ω. f is called an m^*-measurable mapping if the set $f^{-1}(U)$ is m^*-measurable for every open set $U\subseteq\Omega$.

Definition 2. Let G be a topological space. Let f be a mapping of G into a topological space Ω. f is called a mapping which has the property of Baire if the set $f^{-1}(U)$ has the property of Baire for every open set $U\subseteq\Omega$.

Definition 3. Let G be a topological group. G is called to be σ-bounded, if for every open set $U\subseteq G$ there exists a sequence $a_1, a_2,\cdots, a_n,\cdots$ of elements of G such that $G=\bigcup_{i=1}^{\infty} a_i U$.

Theorem 1. Let G be a locally compact group and m^* a left-invariant Haar's outer measure in G. If f is an m^*-measurable homomorphic mapping of G into a σ-bounded topological group G^*, then f is continuous.

Proof. Let $H^*=f(G)$. If we introduce the relative topology in H^*, then H^* becomes a σ-bounded topological group. For the proof of our theorem it is sufficient to show that f is a continuous mapping of G into H^*. Let U^* be an arbitrary neighborhood of the identity