82. On a Theorem of W. Sierpiński and S. Ruziewicz

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In my Note [1], we have generalized a theorem of W. Sierpiński [3]. In this Note we shall prove a theorem of S. Ruziewicz [2] and consider the relation of my result and his theorem. My result [1] is stated as follows: *Let M be an ordered set with power m. For a power n, $n\geq m$, if and only if the following proposition is true: for every element a of M, we can assign a family $\mathcal{F}(a)$ of intervals such that each interval of it has a as end point and $\overline{\overline{\mathcal{F}(a)}}<n$, and one of any distinct element of M is an end point of an interval of some $\mathcal{F}(a)$.*

For an ordered set M with power m, let us consider the product space $M\times M$, then $A=\{(x, y)\mid y\in\mathcal{F}(x)\}\cup\{(x, x)\mid x\in M\}$ and $B=\{(x, y)\mid x\in\mathcal{F}(y)\}$ are disjoint. Further $A\smile B=M\times M$, therefore the set A, B gives a partition of $M\times M$. Hence the section $A(x_0)$ of A by a given x_0 has the power $<n$. On the other hand, the section $B(y_0)$ of B by any y has the power $<n$. Thus we have the following

Proposition. *Let M be an ordered set with power m. If $m\leq n$, then the product space $M\times M$ is decomposed into two sets A and B such that A meets with power $<n$ on every parallel line to the second coordinate axis and B meets with power $<n$ on every parallel line to the first coordinate axis.*

We shall prove the converse of the proposition. To prove that $m\leq n$, suppose that the set A, B is a partition of $M\times M$, and A, B satisfy the condition mentioned. Then we define $\Phi(a)$ as the set $\{y\mid (a, y)\in A, a\neq y\}\cup\{x\mid (x, a)\in B, x\neq a\}$. Therefore we have $\overline{\overline{\Phi(a)}}<n$, and for each a of M, we may define $\Phi(a)$. If x and y are distinct elements of M, then, by $(x, y)\in M$, $(x, y)\in A$ or $(x, y)\in B$. If $(x, y)\in A$, then $x\in\Phi(y)$, and if $(x,y)\in B$, then $y\in\Phi(x)$. Let us define $\mathcal{F}(a)$ as all intervals (a, x) such that $x\in\Phi(a)$. It is obvious that $\overline{\overline{\mathcal{F}(a)}}<n$, and one of distinct elements is an end point of an interval of type $\mathcal{F}(a)$.

Therefore we have the following

Theorem. *Let M be an ordered set with power m. A power n is not less than m, if and only if the following statement: the product space $M\times M$ is decomposed into two disjoint sets such that one meets with the power $<n$ on each parallel line to the first coordinate axis and the other meets with power $<n$ on each parallel line to the second coordinate axis.*

Such a theorem was stated by S. Ruziewicz [2] and a special