77. *On Convergence Criteria for Fourier Series. I*

By Kenji YANO
Department of Mathematics, Nara Women's University, Nara, Japan

(Comm. by Z. SUETUNA, M.J.A., June 12, 1958)

1. Introduction. Let $\varphi(t)$ be an even function, integrable in $(0, \pi)$, periodic of period 2π, and let

$$\varphi(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt$$

and

$$s_n = \frac{1}{2} a_0 + \sum_{\nu=1}^{n} a_\nu.$$

Hardy and Littlewood [1] proved the following

THEOREM A. If

$$(1.1) \qquad \int_0^t |\varphi(u)-s|\,du = o\left(t/\log\frac{1}{t}\right) \qquad (t\to 0),$$

and if for some positive δ

$$(1.2) \qquad a_n > -An^{-\delta},\ A>0,$$

then $s_n \to s$.

The proof requires a very difficult Tauberian theorem, and so later Szász [2] gave an alternate proof under the additional condition

$$|\varphi(t)-s| < t^{-c},\ c \text{ a positive constant.}$$

Recently, Wang [3] and Sunouchi [4] proved Theorem A by the method of Riesz summability, and the latter's extension is as follows:

THEOREM B (Sunouchi). If

$$\int_0^t |\varphi(u)-s|\,du = o\left(t/f\left(\frac{1}{t}\right)\right) \qquad (t\to 0),$$

and if $a_n > -\mu(n, A)$ for some positive A, then $s_n \to s$, where $f(x)$ and $\mu(x, A)$ are defined by the conditions 1° $f(x)>0$, $f'(x)>0$, 2° $F(x)=\int^x (1/uf(u))\,du \uparrow \infty$ as $x\uparrow\infty$, and 3° $\mu(x, A)=1/F^{-1}(F(x)-A)$.

In this paper, we shall first give an another proof to Theorem A by the method of de la Vallée Poussin summability, and generalize it in alternate form slightly different from Theorem B. In § 3 we refer to jump functions.

THEOREM 1. If (1.1) holds, and if for some positive δ

$$(1.3) \qquad s_{n+\nu}-s_n > -\varepsilon_n \quad \text{for} \quad \nu=1,2,\cdots,[n^\delta],$$

where $\varepsilon_n>0$ and $\varepsilon_n\to 0$, then $s_n\to s$.

Observing that $\varepsilon_n\to 0$ may be as slowly as we wish, Theorem A is a corollary of Theorem 1 since (1.3) holds whenever $a_n > -An^{-\delta-\varepsilon}$, $\varepsilon>0$, which is (1.2) replaced δ by $\delta+\varepsilon$.