93. A Generalization of Morita's Theorem concerning Generalized Uni-serial Algebras

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Recently K. Morita [1] has obtained the following theorem: a finite dimensional (associative) algebra A over a commutative field is generalized uni-serial if and only if every residue class algebra of A is a QF-3 algebra.

In this note we shall establish, by making use of notion of QF-3 rings introduced by H. Tachikawa, a generalization of this theorem for the case of rings with minimum condition for left and right ideals.

1. Throughout this note a ring will be assumed to have a unit element 1 and to satisfy the minimum (whence the maximum) condition for left and right ideals.

Following R. M. Thrall [2] we shall say that for a ring A, a faithful left A-module U is a minimal faithful left A-module if deletion of any direct summand of U leaves non-faithful left A-module. A ring A is said to be a QF-3 ring if it has a unique minimal faithful left A-module.

H. Tachikawa [3] has shown that A is a QF-3 ring if and only if there exists a faithful, projective, injective, left A-module; if A is a QF-3 ring, the direct sum of U_λ $(\lambda=1,\cdots,m)$ is the unique minimal faithful left A-module where $U_1,\cdots,U_m$ are a representative set of primitive left ideals which are injective (and projective).

Now our theorem is stated as follows.

Theorem. *A ring A is generalized uni-serial if and only if every residue class ring of A is a* QF-3 *ring.*

Proof. Suppose that A is generalized uni-serial. Then every residue class ring of A is generalized uni-serial and hence it is a QF-3 ring. This proves the "only if" part.

Conversely, suppose that every residue class ring of A is a QF-3 ring. It is sufficient, by a theorem of T. Nakayama [4], to show that the residue class ring A/N^2 is generalized uni-serial, where N is the radical of A. Hence we have only to prove the following lemma.

Lemma. *Let A be a* QF-3 *ring such that* $N^2=0$ *where N is the radical of A. Then A is generalized uni-serial.*

2. *Proof of the lemma.* As is well known, we have a direct-sum decomposition of A into indecomposable left [right] ideals: