118. Notes on Lattices

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Let L be a lattice with an inclusion relation $\leqq$, meet $a\frown b$ and join $a\smile b$. L. M. Blumenthal and D. O. Ellis [2] showed that the follwing three relations (G), (G*) and (G**) are equivalent in modular lattices, and that they are also equivalent to metric betweeness for normed lattices.

(G) $\quad (a\frown c)\smile(b\frown c)=c=(a\smile c)\frown(b\smile c)$

(G*) $\quad (a\frown c)\smile(b\frown c)=c=c\smile(a\frown b)$

(G**) $\quad (a\smile c)\frown(b\smile c)=c=c\frown(a\smile b)$

Recently, Y. Matsushima [3] introduced for any lattice L three kinds of sets in L as follows:*)

$$J(a,b)=\{x\mid x=(a\frown x)\smile(b\frown x)\}$$
$$CJ(a,b)=\{x\mid x=(a\smile x)\frown(b\smile x)\}$$
$$B(a,b)=J(a,b)\wedge CJ(a,b).$$

He gave among others a characterization of distributive lattices by using $B(a,b)$, and a characterization of modular lattices by using $B(a,b)$ and $B^*(a,b)$ in [3, 4].

In this note we give some characterizations of modular lattices by $J(a,b)$ and $CJ(a,b)$, which also imply that (G), (G*) and (G**) are equivalent only in modular lattices. We also give two characterizations of distributive lattices by using $J(a,b)$ and $CJ(a,b)$ respectively, each of which is the dual of the other.

LEMMA 1. *If (a,b) is a modular pair [1, p. 100], then $[a\frown b,b]$ is contained in $CJ(a,b)$.*

PROOF. Choose x from $[a\frown b,b]$; then $x\leqq b$ and $(x\smile a)\frown(x\smile b)=(x\smile a)\frown b=x\smile(a\frown b)$ since (a,b) is a modular pair. While $a\frown b\leqq x$, we have $(x\smile a)\frown(x\smile b)=x$. This shows that $[a\frown b,b]\subset CJ(a,b)$.

LEMMA 2. *If $[a\frown b,b]$ is contained in $CJ(a,b)$, then (a,b) is a modular pair.*

PROOF. Let $x\leqq b$, and consider $x\smile(a\frown b)$. Then $a\frown b\leqq x\smile(a\frown b)\leqq b$ and hence by assumption $x\smile(a\frown b)\in CJ(a,b)$. Hence $x\smile(a\frown b)=(x\smile(a\frown b)\smile a)\frown(x\smile(a\frown b)\smile b)=(x\smile a)\frown(x\smile b)=(x\smile a)\frown b$. This shows that (a,b) is a modular pair.

LEMMA 2′. *If $[b,a\smile b]\subset J(a,b)$ for any two elements a and b, then L is a modular lattice.*

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*) We denote the set-theoretical inclusion and intersection by $\subset$ and $\wedge$. We also use $[a)$, $(a]$ and $[a,b]$ for $\{x\mid a\leqq x\}$, $\{x\mid x\leqq a\}$ and $\{x\mid a\leqq x\leqq b\}$ respectively.