110. On Determination of the Class of Saturation in the Theory of Approximation of Functions

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1. **Introduction.** Let $f(x)$ be an integrable function, with period 2π and let its Fourier series be

$$(1)\qquad \frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)\equiv\sum_{k=0}^{\infty}A_k(x).$$

Let $g_k(n)$ $k=1,2,\cdots$ be the summating function and consider a family of transforms of (1) of a summability method G,

$$(2)\qquad P_n(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}g_k(n)(a_k\cos kx+b_k\sin kx)$$

where the parameter n needs not be discrete.

If there are a positive non-increasing function $\varphi(n)$ and a class K of functions in such a way that

(I) $\|f(x)-P_n(x)\|=o(\varphi(n))$[1] implies $f(x)=$constant;

(II) $\|f(x)-P_n(x)\|=O(\varphi(n))$ implies $f(x)\in K$;

(III) for every $f(x)\in K$, one has $\|f(x)-P_n(x)\|=O(\varphi(n))$,

then it is said that the method of summation G is saturated with order $\varphi(n)$ and its class of saturation is K. This definition is due to J. Favard [2].

The purpose of this article is to determine the order and the class of saturation for several familiar summation methods. M. Zamansky [5] has solved this problem for the method of Cesàro-Fejér, with respect to the space (C) of continuous functions; P. L. Butzer [1] studied the cases of methods of Abel-Poisson and Gauss-Weierstrass, employing the theory of semi-groups, but, as he made use of the regularity of the spaces (L^p) $p>1$, he left the question open for the spaces (C) and (L).

We give here a direct method to determine the class of saturation for general method of summability, with respect to the spaces (C) and (L^p) $p\geqq 1$. The above condition (I) is easily verified and the condition (III) is proved by so-called singular integral method. The inverse problem (II) is the key point of this paper.

2. **The inverse problem.** Let us write $\Delta_n(x)=f(x)-P_n(x)$ and suppose that there are positive constants c, r and ρ such that

$$(3)\qquad \lim_{n\to\infty} n^r(1-g_k(n))=ck^{\rho}\text{ [2]}\qquad (k=1,2,\cdots).$$

1) The norm means (C)- or (L^p)-$(p\geqq 1)$ norm.

2) To fix the ideas, we take the limit as $n\to\infty$; but, as is easily seen, the following arguments remain valid, with appropriate modifications, in other cases (see Theorem 2 below).