5. *On Approximation of Quasi-conformal Mapping*

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In this short note we are concerned with approximation to the general (not necessarily differentiable) quasi-conformal mapping by means of the smooth ones under the condition that the correspondence of a finite number of boundary points shall remain fixed.

In the course of our proof Ahlfors existence theorem plays an important rôle. The notations employed here for convenience are as follows:

$\mathfrak{F}$: The class of all the quasi-conformal mappings between the upper half-planes,

$M(z;\rho;g)$: Areal mean of an integrable function $g(z)$ over the disk $|\zeta-z|\leq\rho$, i.e.

$$M(z;\rho;g)=\frac{1}{\pi\rho^2}\int_0^\rho\int_0^{2\pi}g(z+re^{i\theta})rd\theta\,dr.$$

Proposition. Let $w=f(z)$ *be a quasi-conformal mapping in Pfluger-Ahlfors sense which is a homeomorphism between* $\Im z>0$ *and* $\Im w>0$. *Let* $x_1<x_2<\cdots<x_{k-1}<x_k$ *be points on* $\Im z=0$ *and* $f(x_\nu)=u_\nu$ $(\nu=1,2,\cdots,k)$. *Then there exists a sequence* $\{f_n(z)\}$ *of quasi-conformal mappings* C^1 *between* $\Im z>0$ *and* $\Im w>0$, *such that* $f_n(z)$ *converges to* $f(z)$ *uniformly in* $\Im z>0$ *as* $n\to\infty$ *with the condition* $f_n(x_\nu)=u_\nu$ $(\nu=1,2,\cdots,k)$ *and* $|\partial f_n/\partial z|$ *has a positive lower bound depending only on* n.

Proof. Mathematical induction with respect to the number of distinguished boundary points is available.

1) We first show that the proposition is true in case $k=3$. We may assume without loss of generality $x_1=u_1=-\infty$, $x_2=u_2=-1$, $x_3=u_3=0$.

Let $\{R_n\}$ and $\{\varepsilon_n\}$ be two sequences of positive numbers such that $R_n\uparrow\infty$ and $\varepsilon_n\downarrow 0$ respectively as $n\to\infty$. Let D_n be the domain which is the intersection of the disk $|z|<R_n$ and the half-plane $\Im z>\varepsilon_n$. We approximate the eccentricity $h(z)=\frac{\partial f}{\partial\bar{z}}\Big/\frac{\partial f}{\partial z}$ of the given mapping $w=f(z)$ by a sequence of functions $h_n(z)$ $(n=1,2,\cdots)$ which satisfies the following conditions:

i) $$h_n(z)=\begin{cases}h(z) & z\in D_n\\ 0 & |z|\geq R_n+1,\end{cases}$$

ii) $$h_n(\bar{z})=\overline{h_n(z)},$$

iii) $h_n(z)$ fulfils the Hölder condition of order α $(0<\alpha\leq 1)$ for $|z|<\infty$,