4. *On a Theorem on Modular Lattices*

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1. It is well known that an irreducible, complete, (upper and lower) continuous, complemented modular lattice L is finite-dimensional if and only if the following condition is satisfied:[1]

Condition Δ. L contains no infinite sequence (a_i) of nonzero elements a_i, $i=1,2,\cdots$, such that for every $i>1$ there exists an element b_i satisfying $a_{i-1}\geq a_i\dot{\cup} b_i$[2] *and $a_i\approx b_i$.*

The purpose of the present paper is to prove the following theorem. By $m(L)$ we denote the least upper bound of all integers r such that L contains an independent system of mutually projective nonzero r elements.

Theorem. For any complete upper continuous modular lattice L the condition Δ is equivalent to each of the following two conditions:

Condition M. *$m(L)$ is finite.*

Condition F. *There is no independent countable subset (a_i) such that $a_i\gtrsim a_{i+1}\neq 0$ for every i.*[3]

As a consequence of this we shall obtain

Corollary 1. *Let $\mathfrak{R}$ be a semisimple ring with unit element and assume that $\mathfrak{R}$-left (-right) module $\mathfrak{R}$ is injective. Then $\mathfrak{R}$ is a regular ring (in the sense of v. Neumann), and the following three conditions are equivalent:*

(i) *$\mathfrak{R}$ is of bounded index.*

(ii) *$\mathfrak{R}/\mathfrak{P}$ is a simple ring with minimum condition for every primitive ideal $\mathfrak{P}$.*

(iii) *$\mathfrak{R}$ is P-soluble.*[4]

In this case, $\mathfrak{R}$-right (-left) module $\mathfrak{R}$ is also injective.

2. Henceforth L always will denote a modular lattice with zero.

Lemma 1. Let $a\cap b=a\cap c=0$ and $a\cup b\geq c$. Then $(a\cup c)\cap b\sim_a c$.[5]

Lemma 2. If $0\neq a\leq b=b_1\dot{\cup} b_2\dot{\cup}\cdots\dot{\cup} b_n$, then there exist nonzero a', b' such that $a\geq a'\sim b'\leq b_i$ for some i.

In fact, if $a\cap(b_2\cup\cdots\cup b_n)=0$, then $b_1\cap(a\cup b_2\cup\cdots\cup b_n)\sim a$ by Lemma 1; hence Lemma 2 follows by induction.

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1) See [7].

2) $\dot{\cup}$ denotes the join of independent elements.

3) By $a\gtrsim b$ we mean the existence of c such that $a\geq c\approx b$.

4) See [5].

5) $b\sim_a c$ is meant that $a\dot{\cup} b=a\dot{\cup} c$.