# 18. *Representation of Some Topological Algebras. II*

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**4. Idempotents of rank 1.** This section is devoted to note several fundamental statements concerning the idempotents in an algebra, which we shall need in what follows.

LEMMA 1.[1] *Let $p$ be an idempotent in an algebra $E$. If $Ep$ (resp. $pE$) is a minimal left (resp. minimal right) ideal of $E$, then $pEp$ is a division algebra.*

Proof. It will suffice to prove the lemma in the case of $Ep$ under the assumption that $pEp \neq \{0\}$. Since $p$ is an idempotent, $p$ is the identity in the algebra $pEp$. Let $x$ be a non-zero element in $pEp$; then $Ex$ contains $px=x$, so that $Ex=Ep$ since $Ep$ is a minimal left ideal. It follows that $pEx=pEp$, and hence we have $pEpx=pEp$. Therefore the element $x$ has a left inverse in $pEp$.

LEMMA 2. *Let $E$ be an algebra satisfying the condition* (ii),[2] *and let $p$ be a non-zero idempotent in $E$. If $pEp$ is a division algebra, then $Ep$ is a minimal left ideal and $pE$ is a minimal right ideal of $E$.*

Proof. Let $I$ be a proper non-zero left ideal contained in $Ep$, and $a$ be a non-zero element in $I$. Then by the condition (ii) we can find an element $u \in E$ such that $pua \neq 0$. Since $pua$ is contained in the division algebra $pEp$, it has an inverse $pxp$ in $pEp$; then $pxpua=p$. Therefore the left ideal $I$ contains the element $p$, and so $I$ coincides with $Ep$ contrary to the assumption. Similarly we can prove that $pE$ is a minimal right ideal.

LEMMA 3. *Let $p$ be an idempotent in a Hausdorff topological algebra $E$, and $A$ be a closed subset of $E$. If $Ap$ (resp. $pA$) is contained in $A$, then the set $Ap$ (resp. $pA$) is closed.*

Proof. It will suffice to show that $Ap$ is closed, under the assumption that $Ap \subseteq A$. Let $\mathfrak{F}$ be a filter on the set $Ap$ which converges to an element $a \in A$. Then, since each element of the filter $\mathfrak{F}$ is a subset of the set $Ap$, we have $\mathfrak{F}p=\{Bp;\ B\in\mathfrak{F}\}=\mathfrak{F}$. On the other hand, because of the continuity of the ring multiplication, the filter base $\mathfrak{F}p$ converges to $ap$, and so we have $a=ap$ since $E$ is a Hausdorff space.

1) This lemma is essentially known, but we give a proof for the sake of completeness.

2) Cf. S. Kasahara: Representation of some topological algebras. I, Proc. Japan Acad., **34**, 355–360 (1958).