15. *Some Properties of F-spaces*

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X[1] is called an *F-space* provided for any $f\in C(X)$, $P(f)=\{x; f(x)>0\}$ and $N(f)=\{x; f(x)<0\}$ are completely separated. X has the *F_σ-property* if the closure of any F_σ-open subset of X is open. X has the *E_σ-property* if any $f\in B(U)$ has a continuous extension over X where U is any F_σ-open subset of X. Gillman and Henriksen [1] have proved the interest results on F-spaces; for instance, i) X is σ-complete if and only if for any $f\in C(X)$, $\overline{P(f)}$ is open; ii) X is an F-space if and only if any $f\in B(X-N)$ has a continuous extension over X where N is any Z-set of X. In general, 1) if X has the F_σ-property, X is σ-complete [3] and 2) if X has the E_σ-property, X is an F-space. If X is normal the converses of the above two statements are true [3].

In §1 we shall study the relations between a given space X and its Čech compactification ($=\beta X$) concerning the F_σ-prop., E_σ-prop., σ-completeness, or the property of being an F-space. In §2 we shall consider some questions arising in connection with the theorems in §1.

1. Theorem 1. *The following conditions are equivalent for any space X: 1) X has the F_σ-property; 2) any subspace Y of βX containing X as a proper subset has the F_σ-property; 3) any proper F_σ-open subset of X has the F_σ-property.*

Proof. ($1\to 2$). Let V be any F_σ-open subset of Y. $U=V\frown X$ is also F_σ-open in X and hence $\overline{U}(\text{in } X)$ is open in X. On the other hand, $\beta X=\beta(\overline{U}(\text{in } X))\smile\beta(X-\overline{U}(\text{in } X))$, $\beta(\overline{U}(\text{in } X))\frown\beta(X-\overline{U}(\text{in } X))=\theta$ and $\overline{U}(\text{in } \beta X)=\beta(\overline{U}(\text{in } X))$. Since X is dense in Y and $U=X\frown V$ and V is open in Y, we have $\overline{V}(\text{in } Y)=\overline{U}(\text{in } Y)=\overline{U}(\text{in } \beta X)\frown Y$ and hence $\overline{V}(\text{in } Y)$ is open.

($2\to 3$). Let U be a proper F_σ-open subset of X and let V be F_σ-open in U. V is F_σ-open in X and we put $Y=(\beta X-(\overline{V}(\text{in } \beta X)-V))\smile X$. Since V is F_σ-open in Y and Y has the F_σ-property, $\overline{V}(\text{in } Y)$ is open in Y and hence $\overline{V}(\text{in } U)=\overline{V}(\text{in } Y)\frown U$ is open in U.

($3\to 1$). Let U be any proper F_σ-open subset of X. Suppose that $\overline{U}\neq X$ and $a\in X-\overline{U}$. There exists $f\in B(X)$ such that $f(a)=0$ and

1) A space X considered here is always a completely regular T_1-space. The functions are assumed to be real-valued and $C(X)(B(X))$ denotes the totality of (bounded) continuous functions defined on X.