

25. On Pseudo-compact Spaces

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Notations. Let S be a topological space. Let Z be the family of all sequences $\{f_n(x)\}_{n=1,2,\dots}$, where f_n are (finite real) continuous functions on S such that $f_n(x) \rightarrow 0$ for each $x \in S$. Let Z_0 be the family of all bounded sequences $\{f_n\} \in Z$; let N (resp. E , resp. U) be the family of all non-increasing (resp. equi-continuous, resp. uniformly convergent) sequences $\{f_n\} \in Z$. Further we put $N_0 = N \cap Z_0$, $E_0 = E \cap Z_0$.

If b is a real number, we write $b_+ = \max(b, 0)$.

Lemma 1. $U \cup N \subset E$.

Proof. If $\{f_n\} \in U \cup N$, $x \in S$, $\varepsilon = 2\eta > 0$, then there exist an index p and a neighbourhood V of x such that $|f_n(y)| < \eta$ for each $n > p$ and each $y \in V$. Further we can find a neighbourhood W of x such that $|f_n(x) - f_n(y)| < \varepsilon$ for $n = 1, \dots, p$ and for each $y \in W$. Obviously $|f_n(x) - f_n(y)| < \varepsilon$ for each n and each $y \in V \cap W$.

Lemma 2. If $\{f_n\} \in E$, then the function $f(x) = \sum_{n=1}^{\infty} (|f_n(x)| - \varepsilon)_+$ is continuous for each $\varepsilon > 0$.

Proof. Suppose that $x \in S$ and that $\varepsilon = 2\eta > 0$. There exist an index p and a neighbourhood V of x such that $|f_n(x)| < \eta$ for each $n > p$ and that $|f_n(x) - f_n(y)| < \eta$ for each $y \in V$ and each n . Now, if $n > p$ and if $y \in V$, we have $|f_n(y)| < \varepsilon$, whence $(|f_n(y)| - \varepsilon)_+ = 0$. It follows that $f(y) = \sum_{n=1}^p (|f_n(y)| - \varepsilon)_+$ for each $y \in V$, which completes the proof.

Lemma 3. If S is pseudo-compact, then $N \subset U$.

Proof. If $\{f_n\} \in N$ and if $\varepsilon = 2\eta > 0$, then, by Lemmas 1 and 2, the function $f(x) = \sum_{n=1}^{\infty} (f_n(x) - \eta)_+$ is continuous. Since S is pseudo-compact, there exists a number A such that $f(x) < A$ for each $x \in S$. If $f_n(x) \geq \varepsilon$, then $(f_k(x) - \eta)_+ \geq \eta$ for $k = 1, 2, \dots, n$, so that $n\eta \leq f(x) < A$, $n < A\eta^{-1}$. We see that $f_n(x) < \varepsilon$ for each $n \geq A\eta^{-1}$ and for each $x \in S$.

Lemma 4. If S is pseudo-compact, then $E \subset U$.

Proof. Let $\{f_n\}$ be a sequence of E and let $\varepsilon = 2\eta$ be a positive number. Lemma 2 implies that the functions $g_n(x) = \sum_{k=n}^{\infty} (|f_k(x)| - \eta)_+$ are continuous; obviously $\{g_n\} \in N$ and so, by Lemma 3, $\{g_n\} \in U$. There exists an index p such that $g_p(x) < \eta$ for each $x \in S$. If $n \geq p$, then we have $|f_n(x)| \leq (|f_n(x)| - \eta)_+ + \eta \leq g_p(x) + \eta < 2\eta = \varepsilon$ for each $x \in S$, which proves the lemma.

Lemma 5. If $N_0 \subset U$, then S is pseudo-compact.