51. On Extreme Elements in Lattices

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In a series of papers [2–6] we have studied the concept of *B-covers* and *B*-covers* in lattices. $B(a,b)=\{x \mid axb\}$, $B^*(a,b)=\{y \mid aby\}$, where axb means that $x=(a\smile x)\frown(b\smile x)=(a\frown x)\smile(b\frown x)$. $B(a,b)$ is called the *B-cover* of a and b. We shall say that an element e of a lattice L is an *extreme element* to an element x of L (or e is *extreme* to x) if $B^*(x,e)=e$. An element e is called *extreme* if it is *extreme* to some element of L. By $(a,b)E$ we shall mean that b is *extreme* to a (that is, $B^*(a,b)=b$). We shall call (a,b) an *extreme pair* when $(a,b)E$ and $(b,a)E$; we denote it by $(a,b)E_s$.

If $(a,b)E_s$ and a and b are comparable, then (a,b) equals (O,I). Elements O, I satisfying OxI for all x are called "*extreme*" by G. Birkhoff [1]. If a and b are complemented, then $(a,b)E_s$ by our definition [Theorem 1]. In Theorem 2, we shall give a representation of a Boolean algebra by *maximal extreme B-covers*. If $(a,b)E$, then we shall be able to find out an *extreme pair* $(a_n,b)E_s$ by Theorem 4. If the space of a topological lattice is compact, then we shall call this space a *compact lattice*. After Birkhoff [1], a chain is complete if and only if it is topologically compact. If we denote by $E(a)$ the set of all elements which are extreme to an element a in a compact lattice, then we shall find some interesting properties of $E(a)$ [Theorems 7 and 8], and we shall prove that a *compact extreme lattice* which consists of *extreme elements* is a complemented lattice [Theorem 9].

Theorem 1. *If a and a' are complemented in a lattice, then $(a,a')E_s$.*

Proof. If $aa'x$, then we have $a'=(a\frown a')\smile(a'\frown x)=a'\frown x$, $a'=(a\smile a')\frown(a'\smile x)=a'\smile x$ from $a\frown a'=0$, $a\smile a'=I$, and hence we have $a'=x$, thus we have $B^*(a,a')=a'$. Similarly we have $B^*(a',a)=a$. Hence we have $(a,a')E_s$. The converse of this theorem is not always true.

Lemma 1. *$B(a,b)=B(a\frown b, a\smile b)$ in a distributive lattice.*

Proof. This is proved by [3, Theorem 3].

Lemma 2. *In a Boolean algebra L, if $(a,b)E_s$, then $a\smile b=I$, $a\frown b=O$.*

Proof. Let a' be the complement of a, then $B(a,a')=B(a\frown a', a\smile a')=B(O,I)=L$ by Lemma 1. Hence $b\in B(a,a')$, that is, aba'.