44. *Notes on Uniform Convergence of Trigonometrical Series. I*

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(Comm. by Z. SUETUNA, M.J.A., May 7, 1959)

1. In the preceding paper [1] we have studied the uniform convergence of the series

$$\sum_{n=1}^{\infty}\frac{s_n}{n}\sin nt$$

concerning the Riemann summability (R_1). In this paper we shall treat the cosine-analogue.

Let $\{s_n;\ n=1,2,\cdots\}$ be a sequence with real terms, and let

$$s_n^{\gamma}=\sum_{\nu=0}^{n}A_{n-\nu}^{\gamma-1}s_\nu \qquad (-\infty<\gamma<\infty),$$

where $s_0=0$ and $A_n^{\gamma}=\binom{\gamma+n}{n}$. The theorem to be proved is as follows:

THEOREM 1. Suppose that $0<r$, $0<s<1$ (or $s=1,2,\cdots$), and $0<\alpha\leqq 1$, and that

$$\sum_{\nu=1}^{n}|s_\nu^r|=o(n^{1+r\alpha}), \tag{1.1}$$

$$\sum_{\nu=n}^{2n}(|s_\nu^{-s}|-s_\nu^{-s})=O(n^{1-s\alpha}), \tag{1.2}$$

as $n\to\infty$. Then, (I) when $0<\alpha<1$ the series

$$\sum_{n=1}^{\infty}\frac{s_n}{n}\cos nt \tag{1.3}$$

converges uniformly (on the real axis), and (II) when $\alpha=1$ the series (1.3) converges uniformly if and only if $\sum n^{-1}s_n$ converges.

COROLLARY 1. If

$$\sum_{\nu=n}^{2n}(|s_\nu^{-1}|-s_\nu^{-1})=O(1) \qquad (n\to\infty),$$

where $s_n^{-1}=s_n-s_{n-1}$, and if the series in

$$g(t)=\sum_{n=1}^{\infty}s_n\sin nt \tag{1.4}$$

converges boundedly in the interval (δ,π) for any $\delta>0$, then a necessary and sufficient condition for the convergence of the Caucy integral

$$\int_{\to 0}^{\pi}g(t)\,dt \tag{1.5}$$

is the convergence of the series $\sum n^{-1}s_n$.

This is a theorem of Izumi [2, 3].

This corollary follows from Theorem 1 with $r=s=\alpha=1$, since the convergence of the series in (1.4) implies $s_n=o(1)$, cf. Zygmund [4,