61. On Locally Bounded Functions

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Recently some characterisations of countably compact spaces and pseudo-compact spaces were obtained by A. Appert [1], E. Hewitt [3], J. Kersten [4] and J. Marík [5].

Following G. Aquaro [2], we shall define locally boundedness of functions, and we shall prove some results on locally boundedness.

Let S be a topological space, and let $f(x)$ be a real valued finite function (not necessary continuous) on S. A function $f(x)$ is said to be *locally bounded at a point* x_0, if there is a neighbourhood V of x_0 such that $f(x)$ is bounded on V. A function $f(x)$ is said to be *locally bounded* if $f(x)$ is locally bounded at every point of S.

It is clear that any continuous function on S is locally bounded. We shall show the following

Proposition 1. 1) *If S is countably compact, then any locally bounded function on S is bounded.*

2) *If every locally bounded function on S is bounded, then S is pseudo-compact.*

Proof. The second part of Proposition 1 is clear. To prove the first part, we shall suppose that there is a locally bounded and unbounded function $f(x)$. For every positive integer n, the set $A_n=\{x\,|\,|f(x)|\geq n\}$ is not empty, and the sequence of sets $\{A_n\}$ is decreasing. Therefore $\{\bar{A}_n\}_{n=1,2,\dots}$ is a decreasing sequence of closed sets. Hence $\bigcap_{n=1}^{\infty}\bar{A}_n$ is not empty. Let $x_0\in\bigcap_{n=1}^{\infty}\bar{A}_n$, since $f(x)$ is locally bounded at the point x_0, there is a neighbourhood V of x_0 such that $f(x)$ is bounded on V. Hence, $|f(x)|\leq N$ on V for an integer N. On the other hand, by $x_0\in\bigcap_{n=1}^{\infty}\bar{A}_n$, there is a point x' in V such that $|f(x')|>N$. This completes the proof.

Corollary 1. *The following conditions of a normal space S are equivalent:*

1) *S is countably compact.*

2) *S is pseudo-compact.*

3) *Any locally bounded function on S is bounded.*

Corollary 2. *A metric space is compact if and only if every locally bounded function on it is bounded.*

A sequence $\{f_n\}$ of functions on S is said to *converge to f quasi*