84. *On the Sets of Regular Measures. II*

By Hitoshi TSUJIMOTO
University of Osaka Prefecture

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Theorem 5. (1) Let $\nu=\bigcap_{\lambda\in\Lambda}\mu_\lambda$ be the inferior measure of $\{\mu_\lambda\}_{\lambda\in\Lambda}$. Then, if any measurable set E is inner regular with respect to each μ_λ, $\lambda\in\Lambda$ satisfying $\mu_\lambda(E)<\infty$, the measurable set of ν-finite measure is inner regular with respect to ν, too.

(2) Let μ and ν be two measures. Then, if μ is σ-finite and outer (inner) regular, $\nu\leqq\mu$ implies the strictly outer (inner but not necessarily strictly inner) regularity of ν. (These results will be applied, for instance, to the case when ν is the inferior measure of $\{\mu_\lambda\}_{\lambda\in\Lambda}$ and at least one measure μ_λ, $\lambda\in\Lambda$ is σ-finite and outer (inner) regular.)

Proof. (1) If $\nu(E)<\infty$, there exist (refer to (1) of Theorem 4) a sequence, $\{\lambda_i\}_{i=1}^{\infty}$, and a partition $\{A_i\}_{i=1}^{\infty}$ of E such that $\lambda_i\in\Lambda$ $(i=1,2,\cdots)$, $\bigcup_{i=1}^{\infty}A_i=E$, $A_j\frown A_k=\theta$ $(j\neq k)$, $A_i\in S$ $(i=1,2,\cdots)$ and $\nu(E)\leqq\mu_{\lambda_1}(A_1)+\mu_{\lambda_2}(A_2)+\cdots+\mu_{\lambda_i}(A_i)+\cdots<\infty$. For an arbitrary $\varepsilon>0$, let C_i be a compact measurable set contained in A_i such that $\mu_{\lambda_i}(C_i)>\mu_{\lambda_i}(A_i)-\varepsilon/2^{i+1}$ $(i=1,2,\cdots)$ and let $C=\bigcup_{i=1}^{\infty}C_i$. Then $C\subseteqq E$ and $\nu(E-C)\leqq\nu(A_1-C_1)+\nu(A_2-C_2)+\cdots+\nu(A_i-C_i)+\cdots\leqq\mu_{\lambda_1}(A_1-C_1)+\mu_{\lambda_2}(A_2-C_2)+\cdots+\mu_{\lambda_i}(A_i-C_i)+\cdots<\varepsilon/2$. Therefore $\nu(\bigcup_{i=1}^{N}C_i)>\nu(E)-\varepsilon$ for a suitable integer N.

(2) The assumptions of the σ-finiteness and the outer regularity of μ imply clearly the strictly outer regularity of μ, therefore any measure ν such as $\nu\leqq\mu$ is also naturally strictly outer regular.

Next, suppose that μ is σ-finite and inner regular. In this case, there exists a σ-compact, measurable set $C=\bigcup_{i=1}^{\infty}C_i$ such that $C\subseteqq E$ and $\mu(E-C)<\varepsilon/2$, $\nu(E-C)<\varepsilon/2$ for an arbitrary measurable set E and an arbitrary $\varepsilon>0$.

Now we distinguish two cases:

I. $\nu(E)<\infty$. In this instance, $\nu(C-\bigcup_{i=1}^{N}C_i)<\varepsilon/2$, hence $\nu(E-\bigcup_{i=1}^{N}C_i)<\varepsilon$ for a suitable integer N.

II. $\nu(E)=\infty$. It follows $\nu(C)=\infty$ and there exists an integer N such that $\nu(\bigcup_{i=1}^{N}C_i)>M$ for an arbitrary $M>0$.

Remark 1. The following examples show that situations with respect to outer and inner reguralities are not parallel.

Example 1. This shows the falsity of the more general statement than (2) of Theorem 4: if μ_1 and μ_2 are inner regular, then $\nu=\mu_1\bigcap\mu_2$ is also inner regular.

Let X_1 and X_2 be two non-countable sets such that $X_1\frown X_2=\theta$