

83. A Remark on Monotone Solutions of Differential Equations

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In this Note, we shall consider a system of differential equations:

$$(1) \quad \frac{dx}{dt} = P(t)x + Q(t)$$

where, $x = (x_1, x_2, \dots, x_n)$ is a column vector function of t , and $P(t) = (P_{ij}(t))$, $Q(t) = (Q_{ij}(t))$ are $n \times n$ -matrices. Suppose that these functions of t are defined in some interval $[a, +\infty)$ (a is finite or $-\infty$). Any solution

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

of the system (1) is called to be *monotone for $t \rightarrow \infty$* , if for some T , every function $x_i(t)$ is monotone on the interval $[T, +\infty)$. A matrix function $P(t) = (P_{ij}(t))$ is said to be *integrable on an interval $[a, +\infty)$* , if every function $|P_{ij}(t)|$ is integrable on the interval.

Then we have the following

Proposition. *If matrices $P(t)$ and $Q(t)$ of the differential equation (1) are integrable on an interval $[a, +\infty)$, then any monotone solution $x(t)$ for $t \rightarrow +\infty$ is bounded on the interval and $\lim_{t \rightarrow \infty} x(t)$ exists.*

Such a type of Proposition was discussed by B. P. Demidobitch [1]. Applying his method, we shall prove Proposition directly.

To prove Proposition, we shall suppose that $n=1$ for simplicity. Let $x(t)$ be a monotone solution of (1) for an interval $[a, +\infty)$. Since $P(t)$, $Q(t)$ are integrable, for any $\varepsilon > 0$ ($1 > \varepsilon$), there is a large number T , and then we have

$$\int_{t_1}^{t_2} |P(t)| dt < \varepsilon, \quad \int_{t_1}^{t_2} |Q(t)| dt < \varepsilon$$

for $t_1, t_2 \geq T$. From (1), we can write

$$x(t) = x(T) + \int_T^t P(u)x(u)du + \int_T^t Q(u)du.$$

By the mean theorem of integral calculus, there is an α such that $T \leq \alpha \leq t$ and

$$x(t) = x(T) + x(T) \int_T^\alpha P(u)du + x(t) \int_\alpha^t P(u)du + \int_T^t Q(u)du.$$