76. On Quasi-normed Space. I

By Tomoko KONDA

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Recently a linear metric space which is defined by a quasi-norm was considered by M. Pavel [1] and S. Rolewicz [2]. In this paper, we shall consider such a new linear space with the metric, and establish some results.

Definition 1. *Let E be a linear space over the real field Φ. A real function $\|x\|$ of x is called a quasi-norm with the power r if it satisfies the following conditions.*

1° $\|x+y\|\leq\|x\|+\|y\|$, *for any* $x, y\in E$.

2° $\|\lambda x\|=|\lambda|^r\|x\|$, *for* $\lambda\in\Phi$ *and* $x\in E$, $0<r\leq 1$.

3° $\|x\|=0$ *if and only if* $x=0$.

Let $\|x\|$ be a quasi-norm with the power r and let $d(x,y)=\|x-y\|$, $x\in E$, $y\in E$, then d is distance in E. A linear topological space which is defined by the distance d is called a quasi-normed space with the power r.

Definition 2. *Let E be a quasi-normed space with the power r and if E is complete with the distance d. E will be called a (QN) space with the power r.*

a) By the trivial relations:

$$\|(x+y)-(x_n+y_n)\|\leq\|x-x_n\|+\|y-y_n\|$$

$$\|\lambda x-\lambda_n x_n\|\leq\|\lambda x-\lambda_n x\|+\|\lambda_n x-\lambda_n x_n\|\leq|\lambda-\lambda_n|^r\|x\|+|\lambda_n|^r\|x-x_n\|,$$

if $\lambda_n\to\lambda$, $x_n\to x$ and $y_n\to y$, then we have the convergence $x_n+y_n\to x+y$, $\lambda_n x_n\to\lambda x$. Hence $(x,y)\to x+y$. $(\lambda,x)\to\lambda x$ are continuous on two variables.

b) From the relation:

$$\|x\|=\|y+(x-y)\|\leq\|y\|+\|x-y\|$$

we have

$$\|x\|-\|y\|\leq\|x-y\|.$$

If we replace x and y, then we have

$$\|y\|-\|x\|\leq\|x-y\|.$$

Thus we have the following relation $|\,\|x\|-\|y\|\,|\leq\|x-y\|$.

c) If $x_n\to x$, then $\|x_n\|\to\|x\|$.

This follows from b).

Let E be a topological space, R an equivalent relation on E. We generate the quotient topology on the quotient set E/R. It is the strongest topology in which canonical map φ of E on E/R is continuous. The set E/R, with this topology, is called the quotient space of E by R. If E is a quasi-normed space, then the norm of a coset $\dot{x}$ in a quotient