73. On the Unique Factorization Theorem in Regular Local Rings

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Recently Auslander and Buchsbaum [3] have proved that every regular local ring is a unique factorization ring. This proof depends upon the following result of Nagata [1]: *If every regular local ring of dimension* 3 *is a unique factorization ring, then so is every regular local ring of any dimension* (see [1, pp. 411–413]).

This theorem was proved independently by Zariski [2].

Nagata proved this theorem by using homological method and ideas. The purpose of this paper is to prove anew this theorem by a purely ideal-theoretic method in a simpler way than in [1] and [2].

Let $\mathfrak{O}$ be an n dimensional regular local ring.

Let $\mathfrak{m}=\mathfrak{O}u_1+\mathfrak{O}u_2+\cdots+\mathfrak{O}u_n$ be the maximal ideal of $\mathfrak{O}$, and $\mathfrak{O}'=\mathfrak{O}[X_1, X_2, \cdots, X_n]$ be the polynomial ring over $\mathfrak{O}$. Then $\mathfrak{m}'=\mathfrak{m}[X_1, X_2, \cdots, X_n]$ is a prime ideal of $\mathfrak{O}'$. Let $\mathfrak{O}^*$ be the quotient ring of $\mathfrak{O}'$ with respect to $\mathfrak{m}'$, then $\mathfrak{O}^*$ will be n dimensional regular local ring, and $\mathfrak{m}^*=\mathfrak{O}^*u_1+\mathfrak{O}^*u_2+\cdots+\mathfrak{O}^*u_n$ will be the maximal ideal of $\mathfrak{O}^*$. In the following, we shall use $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{p}$, $\mathfrak{q}$, etc. to denote ideals in $\mathfrak{O}$, and $\mathfrak{a}^*$, $\mathfrak{b}^*$, $\mathfrak{p}^*$, $\mathfrak{q}^*$, etc. to denote ideals in $\mathfrak{O}^*$.

We note the following well-known lemma without proof (see, for example, [4]).

Lemma 1. *We have*

(i) $\mathfrak{O}\frown\mathfrak{O}^*\mathfrak{a}=\mathfrak{a}$.

(ii) *If* $\mathfrak{p}$ *is a prime ideal in* $\mathfrak{O}$, *then so is* $\mathfrak{O}^*\mathfrak{p}$ *in* $\mathfrak{O}^*$, *and if* $\mathfrak{q}$ *is* $\mathfrak{p}$-*primary, then* $\mathfrak{O}^*\mathfrak{q}$ *is* $\mathfrak{O}^*\mathfrak{p}$-*primary. Moreover rank* $\mathfrak{p}=$ *rank* $\mathfrak{O}^*\mathfrak{p}$.

A less familiar lemma is:

Lemma 2. *Let* $v^*=u_1X_1+u_2X_2+\cdots+u_nX_n$, *then* v^* *is an element of a minimal base of* $\mathfrak{m}^*$. *Moreover,* $\mathfrak{O}^*\mathfrak{a}\ni v^*$ *holds if and only if* $\mathfrak{a}=\mathfrak{m}$.

Proof. From $\mathfrak{m}^*=\mathfrak{O}^*u_1+\mathfrak{O}^*u_2+\cdots+\mathfrak{O}^*u_n$ follows the equation $\mathfrak{m}^*=\mathfrak{O}^*v^*+\mathfrak{O}^*u_2+\cdots+\mathfrak{O}^*u_n$. Therefore v^* is an element of a minimal base of $\mathfrak{m}^*$.

Since every element of $\mathfrak{O}^*\mathfrak{a}$ can be expressed in the form $P(x)/Q(x)$, $P(x)\in\mathfrak{a}[X_1, X_2, \cdots, X_n]$, $Q(x)\notin\mathfrak{m}[X_1, X_2, \cdots, X_n]$, $\mathfrak{O}^*\mathfrak{a}\ni v^*$ implies that $\mathfrak{a}[X_1, X_2, \cdots, X_n]\ni v^*$, this means $\mathfrak{a}\ni u_1, u_2, \cdots, u_n$, and thereby completes the proof.