69. A Note on Wedderburn Decompositions of Compact Rings

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(Comm. by Z. SUETUNA, M.J.A., July 13, 1959)

Let R be a topological ring with (Perlis-Jacobson) radical N. We say that R admits a *Wedderburn decomposition* if R can be expressed as the form $R=S+N$ (direct sum of S and N as modules) for some closed semisimple subring S. The subring S is called a *Wedderburn factor* of R. Recently Jans has given a necessary and sufficient condition that a compact ring with open radical admits a Wedderburn decomposition [1, Theorem 1]. In this note, we shall extend Jans' theorem to general compact rings. The proof of our theorem is essentially the same as that of Jans'. As consequences of the theorem, however, Corollaries 1 and 2 in [1] are generalized in the natural form.

The following lemma is readily seen from the proof of Theorem 1 in [1] and Theorem E in [5].

Lemma. *Let R be a compact ring with $(p_1\cdots p_n)x=0$ for every x in R and for fixed distinct primes $p_1,\cdots,p_n$. Then $R=R_{p_1}\oplus\cdots\oplus R_{p_n}$, the ring direct sum of closed ideals $R_{p_i}=\{x\mid x\in R,\ p_ix=0\}$. Moreover, each R_{p_i} has a Wedderburn factor S_{p_i}, and hence R has a Wedderburn factor $S=S_{p_1}\oplus\cdots\oplus S_{p_n}$.*

For brevity we shall say that an element x of a topological ring has the *property* (γ) if for every nucleus (=neighborhood of 0) U of the ring there exist distinct primes $p_1,\cdots,p_k$ such that $(p_1\cdots p_k)x\in U$.

We can now give the following

Theorem. *Let R be a compact ring with the radical N and let e^* be the identity element of the residue class ring $R^*=R/N$. Then R admits a Wedderburn decomposition if and only if e^* can be raised to an element e_1 of R having the property (γ).*

Proof. The "only if" part is clear, since any compact semisimple ring is (algebraically and topologically) isomorphic to a complete ring direct sum (with Tychonoff topology) of finite simple rings [2, Theorem 16].

We are going to prove the converse way. Let e be the idempotent element in the closure of the positive powers of e_1 [3, Lemma 3]. It is clear that e is mapped onto e^* by the natural homomorphism $R\rightarrow R^*$. We shall show that the idempotent e has the property (γ). Suppose that U is an arbitrary compact nucleus in R. Let W be a nucleus such that $W\subset U$, $W\cdot R\subset U$. Since e_1 has the property (γ), there exist distinct primes $p_1,\cdots,p_n$ such that $(p_1\cdots p_n)e_1\in W$. There-