108. Remarks on Pseudo-resolvents and Infinitesimal Generators of Semi-groups

By Tosio KATO
Department of Physics, University of Tokyo

Let $X$ be a Banach space and $E(X)$ the algebra of all bounded linear operators on $X$ to $X$. As is well known, a linear operator $A$ in $X$ is the infinitesimal generator of a semi-group $\{U(t)\}$, $0 < t < \infty$, $U(t) \in E(X)$, if i) $A$ is densely defined, ii) the resolvent $(\lambda I - A)^{-1} \in E(X)$ exists for sufficiently large real $\lambda$ and $\|(\lambda I - A)^{-1}\| = O(\lambda^{-1})$ for $\lambda \to +\infty$ and iii) certain additional conditions are satisfied according to the types of semi-groups considered.\(^1\)

The object of the present note is to point out that i) is a consequence of ii), provided that the underlying space $X$ is locally sequentially weakly compact (abbr. l.s.w.c.). In particular this is the case if $X$ is reflexive.\(^2\) This will be shown below as a consequence of a general theorem on pseudo-resolvents.\(^3\) A pseudo-resolvent $J(\lambda)$ is a function on a subset $D$ of the complex plane to $E(X)$ satisfying the resolvent equation

$$ J(\lambda) - J(\mu) = -(\lambda - \mu)J(\lambda)J(\mu), \quad \lambda, \mu \in D. $$

It follows directly from (1) that all $J(\lambda)$, $\lambda \in D$, have a common null space $N$ and a common range $R$, which will be called respectively the null space and the range of the pseudo-resolvent under consideration. $N$ is a closed subspace of $X$, but $R$ need not be closed; we denote by $[R]$ the closure of $R$. Note that $J(\lambda)$ is a resolvent (of a closed linear operator $A$) if and only if $N = \{0\}$; in this case $R$ coincides with the domain of $A$.

Theorem. Let $J(\lambda)$, $\lambda \in D$, be a pseudo-resolvent with the null space $N$ and the range $R$. Let there be a sequence $\{\lambda_n\}$, $n = 1, 2, \ldots$, such that

$$ \lambda_n \in D, \quad |\lambda_n| \to +\infty, \quad ||\lambda_n J(\lambda_n)|| \leq M = \text{const}. $$

Then we have

$$ N \cap [R] = \{0\}. $$

If, in particular, $X$ is l.s.w.c., then

$$ X = N \oplus [R]. $$


\(^2\) When $X$ is a Hilbert space, this fact was noted by C. Foiaș, Bull. Soc. Math. France, 85, 263 (1957).

\(^3\) Hille and Phillips: Footnote 1), pp. 126 and 183.