108. *Remarks on Pseudo-resolvents and Infinitesimal Generators of Semi-groups*

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Let X be a Banach space and $E(X)$ the algebra of all bounded linear operators on X to X. As is well known, a linear operator A in X is the infinitesimal generator of a semi-group $\{U(t)\}$, $0<t<\infty$, $U(t)\in E(X)$, if i) A is densely defined, ii) the resolvent $(\lambda I-A)^{-1}\in E(X)$ exists for sufficiently large real λ and $\|(\lambda I-A)^{-1}\|=O(\lambda^{-1})$ for $\lambda\to+\infty$ and iii) certain additional conditions are satisfied according to the types of semi-groups considered.[1]

The object of the present note is to point out that i) *is a consequence of* ii), *provided that the underlying space* X *is locally sequentially weakly compact* (abbr. l.s.w.c.). In particular this is the case if X is reflexive.[2] This will be shown below as a consequence of a general theorem on *pseudo-resolvents*.[3] A pseudo-resolvent $J(\lambda)$ is a function on a subset D of the complex plane to $E(X)$ satisfying the *resolvent equation*

$$(1) \qquad J(\lambda)-J(\mu)=-(\lambda-\mu)J(\lambda)J(\mu), \quad \lambda, \mu\in D.$$

It follows directly from (1) that all $J(\lambda)$, $\lambda\in D$, have a common null space N and a common range R, which will be called respectively the null space and the range of the pseudo-resolvent under consideration. N is a closed subspace of X, but R need not be closed; we denote by $[R]$ the closure of R. Note that $J(\lambda)$ is a resolvent (of a closed linear operator A) if and only if $N=\{0\}$; in this case R coincides with the domain of A.

Theorem. *Let* $J(\lambda)$, $\lambda\in D$, *be a pseudo-resolvent with the null space* N *and the range* R. *Let there be a sequence* $\{\lambda_n\}$, $n=1,2,\cdots$, *such that*

$$(2) \qquad \lambda_n\in D, \quad |\lambda_n|\to+\infty, \quad \|\lambda_n J(\lambda_n)\|\leq M=\text{const.}$$

Then we have

$$(3) \qquad N\cap[R]=\{0\}.$$

If, in particular, X *is l.s.w.c., then*

$$(4) \qquad X=N\oplus[R].$$

1) See E. Hille and R. S. Phillips: Functional analysis and semi-groups, Am. Math. Soc. Colloq. Publ., Vol. 31, Theorems 12.3.1, 12.3.2, 12.4.1 and 12.5.1.

2) When X is a Hilbert space, this fact was noted by C. Foiaş, Bull. Soc. Math. France, **85**, 263 (1957).

3) Hille and Phillips: Footnote 1), pp. 126 and 183.