105. A Unique Continuation Theorem of a Parabolic Differential Equation

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1. Introduction. Let G be a convex domain of the euclidean $n+1$-space $R_{t,x}$ $(-\infty<t<+\infty, -\infty<x_i<+\infty$ $(i=1,2,\cdots,n))$, containing a curve C: $\{(t, x_i(t)) \mid t\in[a,b]\}$, where $x_i(t)\in C^2[a,b]$.

Consider real solutions u of an inequality of the following kind:

$$(1.1)\qquad \left|\frac{\partial u(t,x)}{\partial t}-a_{ij}(t,x)\frac{\partial^2 u(t,x)}{\partial x_i\partial x_j}\right|\leqq M\left\{\sum_1^n\left|\frac{\partial u(t,x)}{\partial x_i}\right|+|u(t,x)|\right\}.$$

Here $((a_{ij}(t,x)))$ denotes a positive definite, symmetric matrix of real valued functions $a_{ij}(t,x)\in C^2(G)$, and M a constant.

Our purpose in this note is to prove the following theorem for solutions of (1.1).

Theorem. If u is a solution of (1.1) in the convex domain G and if for any $\alpha>0$,

$$(1.2)\qquad \lim_{r\to 0}\max_{\substack{|x-x(t)|=r\\ t\in[a,b]}}\left\{|u(t,x)|, \left|\frac{\partial u}{\partial t}(t,x)\right|, \left|\frac{\partial u}{\partial x_i}(t,x)\right|, \left|\frac{\partial^2 u}{\partial x_i\partial x_j}\right|\right\}|x-x(t)|^{-\alpha}=0$$

then u vanishes identically in the horizontal component.

The method is based upon the ideas of H. O. Cordes [2] and E. Heinz [3]. The tools used are all elementary, but our proof is somewhat complicated.

2. The Cordes' transformation. Assuming $[a,b]\supset[-\varepsilon, 1+\varepsilon]$ $(\varepsilon>0)$, let $\mathring{A}(t)$ be the positive square root of the matrix $A(t)=((a_{ij}(t,x(t))))$. Let

$$x-x(t)=\mathring{A}(t)\tilde{x}\qquad\text{for}\quad t\in[-\varepsilon,1+\varepsilon],$$

then we may assume that for some $R_1>0$,

a) $a_{ik}(t,\tilde{x})\in C^2([-\varepsilon,1+\varepsilon]\times D_{R_1})$ $(D_{R_1}=\{x \mid |x|\leqq R_1\})$,

b) $a_{ik}(t,0)=\delta_{ik}$,

c) there are positive numbers C_1 and C_2 such that for any real vector $(\xi_1,\xi_2,\cdots,\xi_n)$

$$C_1\sum_1^n\xi_i^2\leqq\sum a_{ij}(t,\tilde{x})\xi_i\xi_j\leqq C_2\sum_1^n\xi_i^2.$$

From (a), (b) and (c) we see the following

Lemma 1. For some $R_2, \tilde{R}_2<R_1$ there is a topological transformation from $[-\varepsilon,1+\varepsilon]\times D_{R_2}$ onto $[-\varepsilon,1+\varepsilon]\times D_{\tilde{R}_2}$:

$$\tilde{y}=\tilde{y}(t,\tilde{x}),\ t=t$$

such that it satisfies the following conditions: