104. On Singular Perturbation of Linear Partial Differential Equations with Constant Coefficients. I

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1. Introduction. Let $(t, x)=(t, x_1,\cdots,x_m)$ be $m+1$ real variables in $t\geqq 0$, $x\in E^m$, where E^m denotes the m-dimensional Euclidean space. Let L_ε be an $r\times r$ matrix of differential operators with constant coefficients depending on a parameter ε

$$L_\varepsilon=\sum_{j=1}^{l} P_j(\partial_x, \varepsilon)\,\partial_t^{j}\,{}^{1)}$$

where $P_j(\xi, \varepsilon)$ are $r\times r$ matrices of polynomials in $\xi=(\xi_1,\cdots,\xi_m)$, whose coefficients depends on $\varepsilon\geqq 0$ *continuously*, and let us consider a system of partial differential equations

(1) $$L_\varepsilon[u]=f(t, x, \varepsilon),$$

where $u=(u_\rho\ \rho\downarrow 1,\cdots,r)$ $f=(f_\rho\ \rho\downarrow 1,\cdots,r)$.[2] We assume that $P_l(\xi,\varepsilon)=P_l(\varepsilon)$ does not contain ξ and

(2) $$\det(P_l(\varepsilon))\neq 0 \text{ for } \varepsilon>0.$$

In this note we are concerned with showing the relationship of (1), as $\varepsilon\downarrow 0$, to a particular solution of a related system (for $\varepsilon=0$)

(1°) $$L_0[u]=f(t, x, 0),$$

especially when L_0 is *degenerated*, i.e.

(2°) $$\det(P_l(0))=0.^{3)}$$

Let C_0^∞ be the set of all on E^m infinite times continuously differentiable complex valued functions with compact carrier. For any $u\in C_0^\infty$ we define the norm $\|u\|_p$ by

(3) $$\|u\|_p^2=\int_{E^m}\sum_{|\nu|\leqq p}|\partial_1^{\nu_1}\cdots\partial_m^{\nu_m}u(x)|^2\,dx,^{4)}\quad (|\nu|=\nu_1+\cdots+\nu_m).$$

The completion of C_0^∞ with respect to the norm (3) will be denoted by H_p. H_p is a kind of Hilbert space. One sees easily

$$H_p\supset H_{p'} \text{ and } \|u\|_p\leqq\|u\|_{p'} \text{ if } p<p'.$$

We set $H_\infty=\bigcap_{p<\infty}H_p$, then H_∞ is a linear topological space with a sequence of semi-norms $\|u\|_p$ $(p=0, 1, 2,\cdots)$ for $u\in H_\infty$. H_∞ is dense in H_p for any p, and C_0^∞ is dense in H_∞ (hence in H_p).

Let $\hat\varphi$ be the Fourier transform of $\varphi\in H_p$,

(4) $$\hat\varphi(\xi)=\frac{1}{\sqrt{2\pi}^{\,m}}\int_{E^m}e^{-i\xi\cdot x}\varphi(x)dx=\mathfrak{F}[\varphi],$$

1) We use ∂_t for ∂/∂_t, and ∂_x for $\partial/\partial x_1,\cdots,\partial/\partial x_m$.

2) $(u_\rho\ \rho\downarrow 1,\cdots,r)$ means the r-dimensional vector (column) with the components $(u_1,\cdots,u_r)$.

3) The condition (2) is not essential in the general consideration.

4) ∂_μ is the abbreviation of $\partial/\partial x_\mu$.