103. A Characteristic Property of L_p-Spaces ($p>1$)

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In the theory of L_p-spaces, $p>1$, the fundamental rôle is played by Hölder's inequality:

$$(1)\qquad \int_0^1 f(t)\,g(t)\,dt \leqq \left(\int_0^1 |f(t)|^p\,dt\right)^{1/p}\left(\int_0^1 |g(t)|^q\,dt\right)^{1/q}$$

where $f(t)\in L_p$, $g(t)\in L_q$ and $q=p/p-1$.

This inequality is usually proved by making use of the following special Young's inequality:

$$(2)\qquad \int_0^1 f(t)\,g(t)\,dt \leqq \frac{1}{p}\int_0^1 f(t)|^p\,dt + \frac{1}{q}\int_0^1 |g(t)|^q\,dt.$$

It is well known that, for the function

$$(3)\qquad g(t)=|f(t)|^{p-1}\,sgn\,f(t)=Tf(t),$$

we get the equality sign in (1). Namely, if the equality holds in (2) for a pair of functions, then for the same pair the equality holds in (1). The purpose of this paper is to show that this property is characteristic for L_p-spaces, $p>1$.

The transformation T in (3) has the following properties:

(i) $x\geqq y\geqq 0$ implies $Tx\geqq Ty\geqq 0$;

(ii) $(Tx)[y]=T([y]x)$ for any projector $[y]$;[1]

(iii) $T(-x)=-Tx$.

A transformation T from a universally continuous semi-ordered linear space R into its conjugate space $\overline{R}$;[2] with the above conditions (i)–(iii) is said to be *conjugately similar*.

A function $\|x\|$ on a universally continuous semi-ordered linear space is called a norm if

(i) $\|x\|\geqq 0$; $\|x\|=0$ implies $x=0$;

(ii) $\|\alpha x\|=|\alpha|\,\|x\|$;

(iii) $\|x+y\|\leqq\|x\|+\|y\|$;

(iv) $x\geqq y\geqq 0$ implies $\|x\|\geqq\|y\|\geqq 0$.

The conjugate norm is defined by

$$\|\overline{x}\|=\sup_{\|x\|\leqq 1}(\overline{x}, x)\quad (x\in R,\ \overline{x}\in\overline{R}).$$

We prove the following

Theorem. Let R be a normed universally continuous semi-ordered

1) $[y]x=\bigcup_{n=1}^{\infty}(x\frown n|y|)$ if $x\geqq 0$ and $[y]x=[y]x^+-[y]x^-$ for any $x\in R$.

2) The conjugate space of a normed semi-ordered linear space is the set of norm-bounded and universally continuous linear functionals. See [1, §31].